Regularity of Fractional Integrals on Product Spaces

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Abstract

Stein-Weiss inequality of fractional integrals is extended into product spaces.

1 Introduction

In 1928, Hardy and Littlewood have established an regularity theorem of fractional integrals in one dimensional Euclidean space [1]. Ten years after that, Sobolev extended this result into higher dimensions [2]. It is well known today as Hardy-Littlewood-Sobolev inequality. The investigation of this inequality in weighted norms begun by Hardy and Littlewood themselves, in one dimensional space where the weights are suitable power functions [1]. This weighted inequality was later extended into higher dimensions by Stein and Weiss [3] and now bears the name of Stein-Weiss inequality.

In the present paper, we give an extension of this classical result [3] by studying so-called *strong fractional integral operators*, satisfying certain characteristics on the product space

$$\mathbb{R}^{\mathbf{N}} ~=~ \mathbb{R}^{\mathbf{N}_1}\times\mathbb{R}^{\mathbf{N}_2}\times\cdots\times\mathbb{R}^{\mathbf{N}_n}. \qquad (1.\ 1)$$

Let

$$0<\alpha_i<\mathbf{N}_i, \qquad i=1,2,\ldots,n \qquad \text{and} \qquad \alpha ~=~ \alpha_1+\alpha_2+\cdots+\alpha_n. \qquad (1.\ 2)$$

The strong fractional integral operator $\mathbf{I}_\alpha$ is defined by

$$\Big(\mathbf{I}_\alpha f\Big)(x) ~=~ \int_{\mathbb{R}^{\mathbf{N}}} f(y)\prod_{i=1}^n\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i} dy \qquad (1.\ 3)$$

whose kernel has singularity appeared on each of the coordinate subspaces. It is clear that

$$\left(\frac{1}{|x|}\right)^{\mathbf{N}-\alpha} ~\lesssim~ \prod_{i=1}^n\left(\frac{1}{|x_i|}\right)^{\mathbf{N}_i-\alpha_i}. \qquad (1.\ 4)$$

The weighted norm regularity theory of such *product operators*, commuting with a multi parameters family of dilations, has seen little in progress since the 1980's after a number of pioneering works accomplished by R. Fefferman and Stein. See [10]-[12] and many references cited there . The area remains largely open for fractional integrals. We hereby consider

$$\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(\omega\mathbf{I}_\alpha f\Big)^q(x)dx\right\}^{\frac{1}{q}} ~\lesssim~ \left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(f\sigma\Big)^p(x)dx\right\}^{\frac{1}{p}} \qquad (1.\ 5)$$

for $f\geq0$ and $1<p\leq q<\infty$, where ω^q and $\sigma^{\frac{p}{1-p}}$ are nonnegative, locally integrable functions.

Let $\mathbf{Q}_i$ to be a cube in $\mathbb{R}^{\mathbf{N}_i}$ for every $i = 1, 2, \ldots, n$ and

$$\mathbf{Q} \ = \ \mathbf{Q}_1 \ \times \ \mathbf{Q}_2 \ \times \ \cdots \ \times \ \mathbf{Q}_n. \tag{1. 6}$$

Let $r \geq 1$ and $1 < p \leq q < \infty$. We say $\omega, \sigma \in \mathbf{A}^{\alpha}_{pqr}$ if

$$\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \omega^{qr}(x) dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} (x) dx \right\}^{\frac{p-1}{pr}} \ < \ \infty \tag{1. 7}$$

for every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$. For $r = 1$, we write $\mathbf{A}^{\alpha}_{pq}$ for $\mathbf{A}^{\alpha}_{pq1}$.

Observe that (1. 7) is a parameter-variant of Muckenhoupt characteristic [8]-[9] , testing over all rectangles $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$. The assertion of $r > 1$ is analogue to Fefferman-Phone's condition, initially introduced for $p = q$. See C. Fefferman [6]. On the other hand, we shall find that $\omega, \sigma \in \mathbf{A}^{\alpha}_{pq}$ is an necessity for the norm inequality to hold in (1. 5). Conversely, the $\mathbf{A}^{\alpha}_{pq}$ characterization is not sufficient in general, unless appropriate side conditions are assumed on ω^q and $\sigma^{\frac{p}{1-p}}$. In such product setting, it is natural to require the side conditions satisfied on every coordinate subspace, uniformly in other variables. In particular, if ω^q and $\sigma^{\frac{p}{1-p}}$ satisfy *product A_∞*-property, then $\mathbf{A}^{\alpha}_{pq}$ is equivalent to $\mathbf{A}^{\alpha}_{pqr}$ for r sufficiently close to 1. Nevertheless, given the weights to be power functions, they are not necessarily locally integrable on the coordinate subspaces. This distinguishes our regularity estimates on the product space, whereas Stein-Weiss inequality stands as the first nontrivial and sharp result.

2 Statement of Main Result

Theorem A: *Let $\gamma, \delta \in \mathbb{R}$. For $1 < p \leq q < \infty$, we have*

$$\left\| \int_{\mathbb{R}^{\mathbf{N}}} f(y) \left(\frac{1}{|x|}\right)^{\gamma} \prod_{i=1}^{n} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i} \left(\frac{1}{|y|}\right)^{\delta} dy \right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \ \lesssim \ \|f\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}$$

if and only if

$$\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx \right\}^{\frac{p-1}{p}} \ < \ \infty$$

for every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$.

Remark 2.1 *In the case of $n = 2$, the theorem above is proved by Sawyer and Wang in [7]. However, the delicate method used there relies on the solvability of a linear system, and cannot be generalized when the number of parameters n exceeds 2.*

In order to prove the theorem, we develop a new framework where the product space is decomposed into an union of Dyadic cones. The consisting partial sum operator defined on each cone is essentially an one-parameter fractional integral operator. Under certain conditions, its norm decays exponentially as the eccentricity of the cone getting large.

Sketch of Proof: In Section 3, we introduce our invented framework and show that every partial sum operator defined on a Dyadic cone satisfies the desired regularity. In Section 4, we give certain characteristic estimates for $\omega, \sigma \in \mathbf{A}^{\alpha}_{pq}$. By letting $\omega(x) = |x|^{-\gamma}$ and $\sigma(x) = |x|^{\delta}$, we find necessary constraints on the indices p, q, γ, δ and α. In particular, we have $\gamma + \delta \geq 0$. In Section 5, we prove Theorem A in the case of $\gamma \geq 0, \delta \leq 0$ and $\gamma \leq 0, \delta \geq 0$, within the philosophy of iterations, by replacing the weights with appropriate product functions, after a decomposition of the operator $\omega \mathbf{I}_\alpha \sigma^{-1}$. This appears to be the original idea of the method applied in [7]. In section 6, we show that for $\gamma > 0, \delta > 0$, it is suffice to prove the theorem for strict subbalanced indices: $\alpha_i/\mathbf{N}_i > 1/p - 1/q,\ i = 1, 2, \ldots, n$. In Section 7, we prove a decaying estimate on the variant of Muckenhoupt characteristic in (1. 7), as $\mathbf{Q}$ varying its eccentricity, for some $r > 1$ sufficiently close to 1 and $\gamma > 0, \delta > 0$ except on some permissible endpoints. The proof can be then completed by interpolations.

3 Cone Decomposition and Eccentricity Summability

Let $t_i, i = 1, 2, \ldots, n$ to be nonnegative integers. The partial sum operator $\Delta_{\mathbf{t}} \mathbf{I}_\alpha$ is defined by

$$\Big(\Delta_{\mathbf{t}} \mathbf{I}_\alpha f\Big)(x) ~=~ \int_{\Gamma_{\mathbf{t}}(x)} f(y) \prod_{i=1}^n \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i} dy \qquad (3.~1)$$

where

$$\Gamma_{\mathbf{t}}(x) ~=~ \left\{ y \in \mathbb{R}^{\mathbf{N}} : 2^{-t_i - 1} \leq \frac{|x_i - y_i|}{|x - y|} \leq 2^{-t_i + 1} \right\}. \qquad (3.~2)$$

Observe that each $\Gamma_{\mathbf{t}}(x)$ is a Dyadic cone centered on $x \in \mathbb{R}^{\mathbf{N}}$ with given dilation $\mathbf{t}$.

In particular, we write

$$\Gamma_o(x) ~=~ \Gamma_{\mathbf{t}}(x) \qquad (3.~3)$$

for $t_1 = t_2 = \cdots = t_n = 0$.

Define the n-parameters dilation

$$\mathbf{t}x ~=~ \Big(2^{-t_1} x_1, 2^{-t_2} x_2, \ldots, 2^{-t_n} x_n\Big). \qquad (3.~4)$$

Let $\mathbf{Q}^{\mathbf{t}}$ to be a dilated variant of $\mathbf{Q}$, such that $|\mathbf{Q}^{\mathbf{t}}_i|^{\frac{1}{\mathbf{N}_i}} = 2^{-t_i} |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}$ for $i = 1, 2, \ldots, n$. We have

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \omega^{qr}(\mathbf{t}x)\, dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} (\mathbf{t}x)\, dx \right\}^{\frac{p-1}{pr}} \\
&= \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}^{\mathbf{t}}|} \int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\, dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}^{\mathbf{t}}|} \int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} (x)\, dx \right\}^{\frac{p-1}{pr}} \\
&= \prod_{i=1}^n 2^{t_i\left(\alpha_i - \frac{\mathbf{N}_i}{p} + \frac{\mathbf{N}_i}{q}\right)} \prod_{i=1}^n |\mathbf{Q}^{\mathbf{t}}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}^{\mathbf{t}}|} \int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\, dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}^{\mathbf{t}}|} \int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} (x)\, dx \right\}^{\frac{p-1}{pr}} .
\end{aligned} \qquad (3.~5)$$

For given $\mathbf{t}$, consider $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$ such that

$$|\mathbf{Q}_\imath|^{\frac{1}{\mathbf{N}_\imath}} = \max_{i\in\{1,2,\ldots,n\}} |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}, \qquad \frac{|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}}{|\mathbf{Q}_\imath|^{\frac{1}{\mathbf{N}_\imath}}} = 2^{-t_i}, \qquad i = 1,2,\ldots,n. \tag{3. 6}$$

Let $1 < p \leq q < \infty$ and $r \geq 1$. For each $\mathbf{t}$ fixed, we define

$$\mathbf{A}^{\alpha}_{pqr}(\mathbf{t}:\omega,\sigma) = \sup \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}} \tag{3. 7}$$

where the supremum is taking over all $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$ satisfying (3. 6).

Let $\omega, \sigma \in \mathbf{A}^{\alpha}_{pqr}$ as in (1. 7). We indeed have

$$\sup_{\mathbf{t}} \mathbf{A}^{\alpha}_{pqr}(\mathbf{t}:\omega,\sigma) < \infty. \tag{3. 8}$$

Suppose that

$$\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}, \qquad |\mathbf{Q}_1|^{\frac{1}{\mathbf{N}_1}} = |\mathbf{Q}_2|^{\frac{1}{\mathbf{N}_2}} = \cdots = |\mathbf{Q}_n|^{\frac{1}{\mathbf{N}_n}}. \tag{3. 9}$$

From (3. 5)-(3. 7), we have

$$\begin{aligned} &\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(\mathbf{t}x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(\mathbf{t}x)\,dx\right\}^{\frac{p-1}{pr}} \\ \lesssim\; &\prod_{i=1}^{n} 2^{t_i\left(\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right)} \mathbf{A}^{\alpha}_{pqr}(\mathbf{t}:\omega,\sigma) \end{aligned} \tag{3. 10}$$

for every given $\mathbf{t}$.

We now recall the weighted inequality for one-parameter fractional integrals proved by Sawyer and Wheeden [9]. Let $r > 1$. For each $\mathbf{t}$ fixed, Theorem 1 in [9] implies

$$\begin{aligned} &\left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} dy\right\}^{q} \omega^{q}(\mathbf{t}x)dx\right\}^{\frac{1}{q}} \\ \lesssim\; &\prod_{i=1}^{n} 2^{t_i\left(\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right)} \mathbf{A}^{\alpha}_{pqr}(\mathbf{t}:\omega,\sigma) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \big(f(x)\big)^{p} \sigma^{p}(\mathbf{t}x)dx\right\}^{\frac{1}{p}} \end{aligned} \tag{3. 11}$$

for $1 < p \leq q < \infty$.

By carrying out the proof of the applied theorem, given in section 2 of [9], we find that the implied constant in (3. 11) depends only on p, q, r, α and $\mathbf{N}$.

By using (3. 5)-(3. 11), we have

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(\Delta_{\mathbf{t}} \mathbf{I}_\alpha f\right)^q (x) \omega^q(x) dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\Gamma_{\mathbf{t}}(x)} f(y) \prod_{i=1}^n \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i} dy\right\}^q \omega^q(x) dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\Gamma_o(x)} f(\mathbf{t}y) \left\{\prod_{i=1}^n 2^{-t_i \mathbf{N}_i} \left(\frac{1}{2^{-t_i}|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i}\right\} dy\right\}^q \omega^q(\mathbf{t}x) \prod_{i=1}^n 2^{-t_i \mathbf{N}_i} dx\right\}^{\frac{1}{q}} \\
&\lesssim \prod_{i=1}^n 2^{-t_i\left(\alpha_i + \frac{\mathbf{N}_i}{q}\right)} \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(\mathbf{t}y) \left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} dy\right\}^q \omega^q(\mathbf{t}x) dx\right\}^{\frac{1}{q}} \qquad (3.\ 12) \\
&\lesssim \prod_{i=1}^n 2^{-t_i\left(\alpha_i + \frac{\mathbf{N}_i}{q}\right)} 2^{t_i\left(\alpha_i - \frac{\mathbf{N}_i}{p} + \frac{\mathbf{N}_i}{q}\right)} \mathbf{A}^\alpha_{pqr}(\mathbf{t} : \omega, \sigma) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^p (\mathbf{t}x) dx\right\}^{\frac{1}{p}} \\
&= \mathbf{A}^\alpha_{pqr}(\mathbf{t} : \omega, \sigma) \prod_{i=1}^n 2^{-t_i\left(\alpha_i + \frac{\mathbf{N}_i}{q}\right)} 2^{t_i\left(\alpha_i - \frac{\mathbf{N}_i}{p} + \frac{\mathbf{N}_i}{q}\right)} \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^p (x) \prod_{i=1}^n 2^{t_i \mathbf{N}_i} dx\right\}^{\frac{1}{p}} \\
&= \mathbf{A}^\alpha_{pqr}(\mathbf{t} : \omega, \sigma) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^p (x) dx\right\}^{\frac{1}{p}}.
\end{aligned}
$$

Observe that for each $\mathbf{t}$ fixed, $\Delta_{\mathbf{t}} \mathbf{I}_\alpha$ is essentially an one-parameter fractional integral operator, and satisfies

$$\left\| \omega \left(\Delta_{\mathbf{t}} \mathbf{I}_\alpha\right) \sigma^{-1} \right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \lesssim \mathbf{A}^\alpha_{pqr}(\mathbf{t} : \omega, \sigma). \qquad (3.\ 13)$$

By applying Minkowski inequality, the norm inequality holds in (1. 5) for $1 < p \leq q < \infty$, provided that $r > 1$ and

$$\sum_{\mathbf{t}} \mathbf{A}^\alpha_{pqr}(\mathbf{t} : \omega, \sigma) < \infty. \qquad (3.\ 14)$$

4 Characteristic Estimates

Let $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$ and $\chi_{\mathbf{Q}}$ to be its characteristic function. Consider

$$f(x) = \chi_{\mathbf{Q}}(x) \sigma(x)^{\frac{1}{1-p}} \in \mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right) \qquad (4.\ 1)$$

provided that $\sigma^{\frac{p}{1-p}}$ is locally integrable.

It follows that

$$\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma^{-1}\right)(y) \prod_{i=1}^n \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i} dy \gtrsim \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - 1} \left\{\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}} (y) dy\right\} \chi_{\mathbf{Q}}(x). \qquad (4.\ 2)$$

Let ω^q to be locally integrable. The norm inequality in (1. 5) together with (4. 2) imply

$$\begin{aligned} &\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-1} \left\{ \int_{\mathbf{Q}} \omega^q(x)dx \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx \right\}^{\frac{p-1}{p}} \\ = &\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^q(x)dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx \right\}^{\frac{p-1}{p}} ~<~ \infty. \end{aligned} \tag{4. 3}$$

Hence that $\omega, \sigma \in \mathbf{A}^\alpha_{pq}$ is necessary. Hölder inequality implies $\mathbf{A}^\alpha_{pqr} \subset \mathbf{A}^\alpha_{pq}$ for $r > 1$ in (1. 7). Consider η to be any nonnegative, measurable function satisfying the *product* A_∞-property. Write $\left(x_i, x_i^\dagger\right) \in \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}^{\mathbf{N}-\mathbf{N}_i}$ for $i = 1, 2, \ldots, n$. By applying the reverse Hölder inequality introduced in chapter V of the book by Stein [5], on each of the coordinate subspaces, we have

$$\begin{aligned} \left\{ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \eta^r(x)dx \right\}^{\frac{1}{r}} ~&=~ \left(\frac{|\mathbf{Q}_1|}{|\mathbf{Q}|}\right)^{\frac{1}{r}} \left\{ \int_{\bigotimes_{i=2}^n \mathbf{Q}_i} \left\{ \frac{1}{|\mathbf{Q}_1|}\int_{\mathbf{Q}_1} \eta^r\left(x_1, x_1^\dagger\right)dx_1 \right\} dx_1^\dagger \right\}^{\frac{1}{r}} \\ &\lesssim~ \left(\frac{|\mathbf{Q}_1|}{|\mathbf{Q}|}\right)^{\frac{1}{r}} \left\{ \int_{\bigotimes_{i=2}^n \mathbf{Q}_i} \left\{ \frac{1}{|\mathbf{Q}_1|}\int_{\mathbf{Q}_1} \eta\left(x_1, x_1^\dagger\right)dx_1 \right\}^r dx_1^\dagger \right\}^{\frac{1}{r}} \\ &\lesssim~ \left(\frac{|\mathbf{Q}_1|}{|\mathbf{Q}|}\right)^{\frac{1}{r}} \frac{1}{|\mathbf{Q}_1|} \left\{ \int_{\mathbf{Q}_1} \left\{ \int_{\bigotimes_{i=2}^n \mathbf{Q}_i} \eta^r\left(x_1, x_1^\dagger\right)dx_1^\dagger \right\}^{\frac{1}{r}} dx_1 \right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{by Minkowski integral inequality} \\ &\qquad\qquad\vdots \\ &\lesssim~ \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \eta(x)dx \end{aligned} \tag{4. 4}$$

for some $r > 1$ and every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$. Therefore, if ω^q and $\sigma^{\frac{p}{1-p}}$ satisfy the *product* A_∞-property, $\mathbf{A}^\alpha_{pq}$ is equivalent to $\mathbf{A}^\alpha_{pqr}$ for r sufficiently close to 1.

By taking into account that $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta$ for $\gamma, \delta \in \mathbb{R}$, their integrabilities require

$$\gamma ~<~ \frac{\mathbf{N}}{q}, \qquad \delta ~<~ \mathbf{N}\left(\frac{p-1}{p}\right). \tag{4. 5}$$

Notice that the power functions are not necessary to satisfy the *product* A_∞-property. Indeed, the A_∞-property satisfied on each coordinate subspace which is uniformly in the variable belonging to the complement subspace respectively require $\gamma < \min\{\mathbf{N}_i, i = 1, 2, \ldots, n\}/q$ and $\delta < \min\{\mathbf{N}_i, i = 1, 2, \ldots, n\}\left(\frac{p-1}{p}\right)$.

Let $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^\delta \in \mathbf{A}^\alpha_{pq}$ where the inequality holds in (4. 3) for every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$. Suppose that $\mathbf{Q}_i$ shrink to a single point in $\mathbb{R}^{\mathbf{N}_i}$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = 1$ for all $j \neq i$. By applying Lebesgue Differentiation Theorem, it is necessary to have

$$\frac{\alpha_i}{\mathbf{N}_i} ~\geq~ \frac{1}{p} - \frac{1}{q}, \qquad i ~=~ 1, 2, \ldots, n. \tag{4. 6}$$

On the other hand, by carrying out an one-parameter dilation estimate in (4. 9), we have

$$\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}} ~=~ \frac{\alpha}{\mathbf{N}}. \tag{4. 7}$$

This together with (4. 6) imply

$$\gamma+\delta ~\geq~ 0. \tag{4. 8}$$

Let $\mathbf{Q}^\lambda$ to be a dilated variant of $\mathbf{Q}$ such that $|\mathbf{Q}_i^\lambda|^{\frac{1}{\mathbf{N}_i}} = \lambda|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}$ for every $i = 1, 2, \ldots, n$.

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}dx\right\}^{\frac{p-1}{p}} \\
&=~ \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}^\lambda|}\int_{\mathbf{Q}^\lambda}\left(\frac{\lambda}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}^\lambda|}\int_{\mathbf{Q}^\lambda}\left(\frac{\lambda}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}dx\right\}^{\frac{p-1}{p}} \qquad (x \longrightarrow x/\lambda) \\
&=~ (\lambda)^{\gamma+\delta-\alpha+\mathbf{N}\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^n |\mathbf{Q}_i^\lambda|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}^\lambda|}\int_{\mathbf{Q}^\lambda}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}^\lambda|}\int_{\mathbf{Q}^\lambda}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}dx\right\}^{\frac{p-1}{p}}.
\end{aligned} \tag{4. 9}$$

Consider $|\mathbf{Q}_1|^{\frac{1}{\mathbf{N}_1}} = |\mathbf{Q}_2|^{\frac{1}{\mathbf{N}_2}} = \cdots = |\mathbf{Q}_n|^{\frac{1}{\mathbf{N}_n}} = 1$. The first line of (4. 9) is bounded from below. Suppose (4. 7) fails, either by letting $\lambda \longrightarrow 0$ or $\lambda \longrightarrow \infty$, the last line of (4. 9) vanishes provided that the inequality holds in (4. 3) for every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$. Therefore the characteristic in (4. 9) is invariant under one-parameter changing of dilations. In the remaining of this section, we consider $\mathbf{Q}$ to be centered on the origin.

Let $\mathbf{S}$ to be any proper subset of $\{1, 2, \ldots, n\}$. Define the truncated cube $\mathbf{Q}_i^\varepsilon = \mathbf{Q}_i \setminus \{|x_i| < \varepsilon\} \subset \mathbb{R}^{\mathbf{N}_i}$ for some $\varepsilon > 0$ and every $i \in \mathbf{S}$. We write $\mathbf{Q}^\varepsilon = \bigotimes_{i\in\mathbf{S}} \mathbf{Q}_i^\varepsilon \times \bigotimes_{i\in\mathbf{S}^c} \mathbf{Q}_i$. Let $0 < \lambda < 1$. We set $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for every $i \in \mathbf{S}$ and $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = \lambda$ for every $i \in \mathbf{S}^c$. Suppose that there exists at least one $i \in \mathbf{S}^c$ such that $\alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) > 0$. We have

$$\begin{aligned}
&\lim_{\lambda\longrightarrow 0} \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}^\varepsilon}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}^\varepsilon}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}dx\right\}^{\frac{p-1}{p}} \\
&=~ \lim_{\lambda\longrightarrow 0} (\lambda)^{\sum_{i\in\mathbf{S}^c}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}^\varepsilon}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}^\varepsilon}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}dx\right\}^{\frac{p-1}{p}} \\
&=~ \left(\lim_{\lambda\longrightarrow 0} (\lambda)^{\sum_{i\in\mathbf{S}^c}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\right) \left\{\int\cdots\int_{\bigotimes_{i\in\mathbf{S}}\mathbf{Q}_i^\varepsilon}\left(\frac{1}{\sum_{i\in\mathbf{S}}|x_i|}\right)^{\gamma q}\prod_{i\in\mathbf{S}}dx_i\right\}^{\frac{1}{q}} \\
&\qquad\qquad \times~ \left\{\int\cdots\int_{\bigotimes_{i\in\mathbf{S}}\mathbf{Q}_i^\varepsilon}\left(\frac{1}{\sum_{i\in\mathbf{S}}|x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)}\prod_{i\in\mathbf{S}}dx_i\right\}^{\frac{p-1}{p}} \\
&=~ 0 \qquad \text{for every} \quad \varepsilon > 0.
\end{aligned} \tag{4. 10}$$

Case One: Consider $\gamma \geq 0, \delta \leq 0$. Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for some $i \in \{1, 2, \ldots, n\}$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = \lambda$ for all other $j \neq i$. Suppose $\alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q}\right) = 0$ for every $j \neq i$. We have

$$\begin{array}{l}\displaystyle \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\delta\left(\frac{p}{p-1}\right)} dx \right\}^{\frac{p-1}{p}} \\[2ex] \displaystyle \gtrsim \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{\lambda + |x_i|} \right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{|x_i|} \right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \right\}^{\frac{p-1}{p}} \qquad (4.\ 11) \\[2ex] \displaystyle \gtrsim \left\{ \int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|} \right)^{\gamma q} dx_i \right\}^{\frac{1}{q}}. \end{array}$$

A direct computation shows

$$\int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|} \right)^{\gamma q} dx_i \ \approx \ \ln\left(\frac{1 + \lambda}{2\lambda} \right) \quad \text{if} \quad \gamma = \frac{\mathbf{N}_i}{q} \qquad (4.\ 12)$$

and

$$\int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|} \right)^{\gamma q} dx_i \ \approx \ \left(\frac{1}{2\lambda} \right)^{\gamma q - \mathbf{N}_i} - \left(\frac{1}{\lambda + 1} \right)^{\gamma q - \mathbf{N}_i} \quad \text{if} \quad \gamma > \frac{\mathbf{N}_i}{q}. \qquad (4.\ 13)$$

From (4. 11)-(4. 13), by letting $\lambda \longrightarrow 0$, we need

$$\gamma \ < \ \frac{\mathbf{N}_i}{q} \qquad (4.\ 14)$$

to satisfy the inequality in (4. 3).

On the other hand, suppose that there exists at least one $j \neq i$ such that $\alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. We have

$$\begin{array}{l}\displaystyle \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\delta\left(\frac{p}{p-1}\right)} dx \right\}^{\frac{p-1}{p}} \\[2ex] \displaystyle \gtrsim (\lambda)^{\sum_{j \neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{\lambda + |x_i|} \right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{|x_i|} \right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \right\}^{\frac{p-1}{p}} \qquad (4.\ 15) \\[2ex] \displaystyle \gtrsim (\lambda)^{\sum_{j \neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{0 < |x_i| \leq \lambda} \left(\frac{1}{\lambda} \right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \\[2ex] \displaystyle \gtrsim (\lambda)^{\frac{\mathbf{N}_i}{q} - \gamma + \sum_{j \neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)}. \end{array}$$

Recall the estimate in (4. 10) and take $\mathbf{S} = \{i\}$. We have (4. 15) converging to zero as $\lambda \longrightarrow 0$. The last line of (4. 15) together with (4. 14) imply

$$\gamma \ < \ \frac{\mathbf{N}_i}{q} + \sum_{j \neq i} \alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q} \right), \qquad i \ = \ 1, 2, \ldots, n. \qquad (4.\ 16)$$

The formula in (4. 7) implies that (4. 16) is equivalent to

$$\alpha_i - \frac{\mathbf{N}_i}{p} \ < \ \delta, \qquad i = 1, 2, \ldots, n. \tag{4. 17}$$

Case Two: Consider $\gamma \leq 0, \delta \geq 0$. Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for some $i \in \{1, 2, \ldots, n\}$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = \lambda$ for all other $j \neq i$. Suppose $\alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right) = 0$ for every $j \neq i$. We have

$$\begin{aligned}
&\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx \right\}^{\frac{p-1}{p}} \\
&\gtrsim \ \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{|x_i|}\right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \right\}^{\frac{p-1}{p}} \qquad\qquad (4.\ 18) \\
&\gtrsim \ \left\{ \int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \right\}^{\frac{p-1}{p}}.
\end{aligned}$$

A direct computation shows

$$\int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \ \approx \ \ln\left(\frac{1+\lambda}{2\lambda}\right) \quad \text{if} \quad \delta = \mathbf{N}_i\left(\frac{p-1}{p}\right) \tag{4. 19}$$

and

$$\int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \ \approx \ \left(\frac{1}{2\lambda}\right)^{\delta\left(\frac{p}{p-1}\right) - \mathbf{N}_i} - \left(\frac{1}{\lambda + 1}\right)^{\delta\left(\frac{p}{p-1}\right) - \mathbf{N}_i} \quad \text{if} \quad \delta \ > \ \mathbf{N}_i\left(\frac{p-1}{p}\right). \tag{4. 20}$$

From (4. 18)-(4. 20), by letting $\lambda \longrightarrow 0$, we need

$$\delta \ < \ \mathbf{N}_i\left(\frac{p-1}{p}\right) \tag{4. 21}$$

to satisfy the inequality in (4. 3).

On the other hand, suppose that there exists at least one $j \neq i$ such that $\alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right) > 0$. We have

$$\begin{aligned}
&\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx \right\}^{\frac{p-1}{p}} \\
&\gtrsim \ \prod_{j \neq i} (\lambda)^{\alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{|x_i|}\right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \right\}^{\frac{p-1}{p}} \qquad (4.\ 22) \\
&\gtrsim \ \prod_{j \neq i} (\lambda)^{\alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{0 < |x_i| \leq \lambda} \left(\frac{1}{\lambda}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \right\}^{\frac{p-1}{p}} \\
&\gtrsim \ (\lambda)^{\left(\frac{p-1}{p}\right)\mathbf{N}_i - \delta + \sum_{j \neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)}.
\end{aligned}$$

Recall the estimate in (4. 10) and take $\mathbf{S} = \{i\}$. We have (4. 22) converging to zero as $\lambda \longrightarrow 0$. The last line of (4. 22) together with (4. 21) imply

$$\delta \;<\; \mathbf{N}_i\left(\frac{p-1}{p}\right)+\sum_{j\neq i}\alpha_j-\mathbf{N}_j\left(\frac{1}{p}-\frac{1}{q}\right), \qquad i=1,2,\ldots,n. \tag{4. 23}$$

The formula in (4. 7) implies that (4. 23) is equivalent to

$$\alpha_i-\mathbf{N}_i\left(\frac{q-1}{q}\right) \;<\; \gamma, \qquad i=1,2,\ldots,n. \tag{4. 24}$$

Case Three: Consider $\gamma > 0, \delta > 0$. Recall from (4. 9). The characteristic in (4. 3) for $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$ is invariant under one-parameter dilations.

Define the subsets $\mathbf{U}$ and $\mathbf{V}$ by

$$\begin{aligned}
\mathbf{U} &= \left\{i\in\{1,2,\ldots,n\} : \alpha_i-\frac{\mathbf{N}_i}{p} \;\geq\; 0\right\},\\
\mathbf{V} &= \left\{i\in\{1,2,\ldots,n\} : \alpha_i-\mathbf{N}_i\left(\frac{q-1}{q}\right) \;\geq\; 0\right\}.
\end{aligned} \tag{4. 25}$$

Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = \lambda^{-1}$ for every $i\in\mathbf{U}$ and $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for all other $i\notin\mathbf{U}$. We have

$$\begin{aligned}
&\prod_{i=1}^{n}|\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)}dx\right\}^{\frac{p-1}{p}}\\
&\gtrsim\;\left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathbf{U}}\alpha_i-\frac{\mathbf{N}_i}{p}}\left\{\int\cdots\int_{\bigotimes_{i\in\mathbf{U}}\mathbf{Q}_i}\left(\frac{1}{1+\sum_{i\in\mathbf{U}}|x_i|}\right)^{\gamma q}\prod_{i\in\mathbf{U}}dx_i\right\}^{\frac{1}{q}}\\
&\qquad\qquad\times\;\left\{\prod_{i\in\mathbf{U}}(\lambda)^{\mathbf{N}_i}\int\cdots\int_{\bigotimes_{i\in\mathbf{U}}\mathbf{Q}_i}(\lambda)^{\delta\left(\frac{p}{p-1}\right)}\prod_{i\in\mathbf{U}}dx_i\right\}^{\frac{p-1}{p}}\\
&\gtrsim\;\left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathbf{U}}\alpha_i-\frac{\mathbf{N}_i}{p}}\left\{\int\cdots\int_{\bigotimes_{i\in\mathbf{U}}0<|x_i|\leq 1}\prod_{i\in\mathbf{U}}dx_i\right\}^{\frac{1}{q}}\left\{\prod_{i\in\mathbf{U}}(\lambda)^{\mathbf{N}_i}\int\cdots\int_{\bigotimes_{i\in\mathbf{U}}\mathbf{Q}_i}(\lambda)^{\delta\left(\frac{p}{p-1}\right)}\prod_{i\in\mathbf{U}}dx_i\right\}^{\frac{p-1}{p}}\\
&\gtrsim\;\left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathbf{U}}\alpha_i-\frac{\mathbf{N}_i}{p}-\delta}.
\end{aligned} \tag{4. 26}$$

By letting $\lambda \longrightarrow 0$, we need

$$\sum_{i\in\mathbf{U}}\alpha_i-\frac{\mathbf{N}_i}{p} \;\leq\; \delta \tag{4. 27}$$

to satisfy the inequality in (4. 3).

In the case of $\mathbf{U} = \{1, 2, \dots, n\}$, since γ satisfies the first strict inequality in (4. 5), the formula in (4. 7) implies

$$\begin{aligned} \delta ~&=~ \frac{\mathbf{N}}{q} - \gamma + \sum_{i=1}^{n} \alpha_i - \frac{\mathbf{N}_i}{p} \\ &>~ \sum_{i=1}^{n} \alpha_i - \frac{\mathbf{N}_i}{p} ~=~ \sum_{i\in\mathbf{U}} \alpha_i - \frac{\mathbf{N}_i}{p}. \end{aligned} \tag{4. 28}$$

Suppose that $\mathbf{U}$ is a proper subset of $\{1, 2, \dots, n\}$ and there exists at least one $i \in \mathbf{U}^c$ such that $\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Recall from (4. 9). By applying the estimate in (4. 10) with $\mathbf{S} = \mathbf{U}$, we have (4. 26) converging to zero as $\lambda \longrightarrow 0$. The last line of (4. 26) further implies

$$\sum_{i\in\mathbf{U}} \alpha_i - \frac{\mathbf{N}_i}{p} ~<~ \delta. \tag{4. 29}$$

Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = \lambda^{-1}$ for every $i \in \mathbf{V}$ and $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for all other $i \notin \mathbf{V}$. We have

$$\begin{aligned} &\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx \right\}^{\frac{p-1}{p}} \\ &\gtrsim~ \left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathbf{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i} \left\{ \prod_{i\in\mathbf{V}} (\lambda)^{\mathbf{N}_i} \int \cdots \int_{\bigotimes_{i\in\mathbf{V}} \mathbf{Q}_i} (\lambda)^{\gamma q} \prod_{i\in\mathbf{V}} dx_i \right\}^{\frac{1}{q}} \\ &\qquad\qquad \times \left\{ \int \cdots \int_{\bigotimes_{i\in\mathbf{V}} \mathbf{Q}_i} \left(\frac{1}{1 + \sum_{i\in\mathbf{V}} |x_i|} \right)^{\delta\left(\frac{p}{p-1}\right)} \prod_{i\in\mathbf{V}} dx_i \right\}^{\frac{p-1}{p}} \\ &\gtrsim~ \left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathbf{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i} \left\{ \prod_{i\in\mathbf{V}} (\lambda)^{\mathbf{N}_i} \int \cdots \int_{\bigotimes_{i\in\mathbf{V}} \mathbf{Q}_i} (\lambda)^{\gamma q} \prod_{i\in\mathbf{V}} dx_i \right\}^{\frac{1}{q}} \left\{ \int \cdots \int_{\bigotimes_{i\in\mathbf{V}} 0<|x_i|\le 1} \prod_{i\in\mathbf{V}} dx_i \right\}^{\frac{p-1}{p}} \\ &\gtrsim~ \left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathbf{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i - \gamma}. \end{aligned} \tag{4. 30}$$

By letting $\lambda \longrightarrow 0$, we need

$$\sum_{i\in\mathbf{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i ~\le~ \gamma. \tag{4. 31}$$

to satisfy the inequality in (4. 3).

In the case of $\mathbf{V} = \{1, 2, \dots, n\}$, since δ satisfies the second strict inequality in (4. 5), the formula in (4. 7) implies

$$\begin{aligned} \gamma ~&=~ \left(\frac{p-1}{p}\right)\mathbf{N} - \delta + \sum_{i=1}^{n} \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) \\ &>~ \sum_{i=1}^{n} \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) ~=~ \sum_{i\in\mathbf{V}} \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right). \end{aligned} \tag{4. 32}$$

Suppose that $\mathbf{V}$ is a proper subset of $\{1,2,\dots,n\}$ and there exists at least one $i \in \mathbf{V}^c$ such that $\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Recall from (4. 9). By applying the estimate in (4. 10) with $\mathbf{S} = \mathbf{V}$, we have (4. 30) converging to zero as $\lambda \longrightarrow 0$. The last line of (4. 30) further implies

$$\sum_{v\in\mathbf{V}} \alpha_v - \left(\frac{q-1}{q}\right)\mathbf{N}_v \ < \ \gamma. \tag{4. 33}$$

Remark 4.1 *For* $\gamma \geq 0, \delta \leq 0$ *and* $\gamma \leq 0, \delta \geq 0$, *we have* γ *and* δ *satisfying the strict inequalities in (4. 16) and (4. 23) respectively. For* $\gamma > 0, \delta > 0$, *we need additional constraints on the indices in order to obtain the strict inequalities in (4. 29) and (4. 33). For example, the strict subbalance indices:* $\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right) > 0$, $i = 1,2,\dots,n$.

5 Iteration Estimates

Let $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta} \in \mathbf{A}^{\alpha}_{pq}$. We write $\rho = \gamma + \delta$. The formulae in (1. 2) and (4. 7) imply

$$\rho \ = \ \alpha - \mathbf{N}\left(\frac{1}{p} - \frac{1}{q}\right) \ = \ \sum_{i=1}^{n} \alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right). \tag{5. 1}$$

Recall from (4. 6). Let

$$\rho \ = \ \rho_1 + \rho_2 + \cdots + \rho_n \tag{5. 2}$$

where

$$\rho_i \ = \ \alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right) \ \geq \ 0, \qquad i = 1,2,\dots,n. \tag{5. 3}$$

By applying Young's inequality, we have

$$\left(\frac{1}{|x|}\right)^{\rho} \ = \ \left(\frac{1}{|x_1|^2 + \cdots + |x_n|^2}\right)^{\frac{\rho_1+\cdots+\rho_n}{2}} \ \lesssim \ \prod_{i=1}^{n}\left(\frac{1}{|x_i|}\right)^{\rho_i} \tag{5. 4}$$

provided that $\rho_i \geq 0$ for every $i = 1,2,\dots,n$.

Case One: Consider $\gamma \geq 0$, $\delta \leq 0$. We write $\eta = -\delta \geq 0$. Recall that δ satisfies the strict inequality in (4. 17). By using (5. 3), we have

$$\begin{aligned} \rho_i + \eta \ &= \ \alpha_i - \mathbf{N}_i\left(\frac{1}{p} + \frac{1}{q}\right) - \delta \\ &= \ \left(\alpha_i - \frac{\mathbf{N}_i}{p} - \delta\right) + \frac{\mathbf{N}_i}{q} \ < \ \frac{\mathbf{N}_i}{q}, \qquad i \ = \ 1,2,\dots,n. \end{aligned} \tag{5. 5}$$

From (5. 3) and (5. 5), the two pairs of weights

$$\omega(x_i) \ = \ \left(\frac{1}{|x_i|}\right)^{\rho_i+\eta}, \ \ \sigma(x_i) \ = \ \left(\frac{1}{|x_i|}\right)^{\eta} \qquad \text{and} \qquad \omega(x_i) \ = \ \left(\frac{1}{|x_i|}\right)^{\rho_i}, \ \ \sigma(x_i) \ = \ 1 \tag{5. 6}$$

both satisfy the sufficient conditions of Stein-Weiss inequality [3], defined on $\mathbb{R}^{\mathbf{N}_i}$ for every $i = 1,2,\dots,n$.

Let $j \in \{1, 2, \dots, n\}$ and write $x = \left(x_j, x_j^\dagger\right) \in \mathbb{R}^{\mathbf{N}_j} \times \mathbb{R}^{\mathbf{N}-\mathbf{N}_j}$. By applying Stein-Weiss inequality [3] on every subspace $\mathbb{R}^{\mathbf{N}_i}, i = 1, 2, \dots, n$, we have

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{|y_j|\sim|y|} f(y)\left(\frac{1}{|x|}\right)^{\rho+\eta}\prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}|y|^{\eta}dy\right\}^{q} dx\right\}^{\frac{1}{q}}\\
\lesssim\ &\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\left(\frac{1}{|x|}\right)^{\rho}\prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{|y_j|}{|x_j|}\right)^{\eta}dy\right\}^{q} dx\right\}^{\frac{1}{q}}\\
\lesssim\ &\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\left\{\prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|x_i|}\right)^{\rho_i}\right\}\left(\frac{|y_j|}{|x_j|}\right)^{\eta}dy\right\}^{q} dx\right\}^{\frac{1}{q}} \quad \text{by (5. 4)}\\
=\ &\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\left\{\prod_{i\neq j}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|x_i|}\right)^{\rho_i}\right\}\left(\frac{1}{|x_j-y_j|}\right)^{\mathbf{N}_j-\alpha_j}\left(\frac{1}{|x_j|}\right)^{\rho_j}\left(\frac{|y_j|}{|x_j|}\right)^{\eta}dy\right\}^{q} dx\right\}^{\frac{1}{q}}\\
\lesssim\ &\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}} f\left(x_j, y_j^\dagger\right)\prod_{i\neq j}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|x_i|}\right)^{\rho_i}dy_j^\dagger\right\}^{p} dx_j\right\}^{\frac{q}{p}} dx_j^\dagger\right\}^{\frac{1}{q}}\\
\lesssim\ &\left\{\int_{\mathbb{R}^{\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}} f\left(x_j, y_j^\dagger\right)\prod_{i\neq j}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|x_i|}\right)^{\rho_i}dy_j^\dagger\right\}^{q} dx_j^\dagger\right\}^{\frac{p}{q}} dx_j\right\}^{\frac{1}{p}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Minkowski integral inequality}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots\\
\lesssim\ &\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left(f(x)\right)^{p}dx\right\}^{\frac{1}{p}} \qquad \text{for } f \in \mathbf{L}^{p}\left(\mathbb{R}^{\mathbf{N}}\right) \text{ and } 1 < p \leq q < \infty.
\end{aligned}
\tag{5. 7}
$$

Case Two: Consider $\gamma \leq 0$, $\delta \geq 0$. We write $\eta = -\gamma \geq 0$. Recall that γ satisfies the strict inequality in (4. 24). By using (5. 3), we have

$$
\begin{aligned}
\rho_i + \eta\ &=\ \alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\gamma\\
&=\ \left(\alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right)-\gamma\right)+\mathbf{N}_i\left(\frac{p-1}{p}\right)\ <\ \mathbf{N}_i\left(\frac{p-1}{p}\right), \qquad i = 1, 2, \dots, n.
\end{aligned}
\tag{5. 8}
$$

Observe that by (5. 3) and (5. 8), the two pairs of weights

$$
\omega(x_i)\ =\ |x_i|^{\eta},\ \ \sigma(x_i)\ =\ |x_i|^{\rho_i+\eta} \qquad \text{and} \qquad \omega(x_i)\ =\ 1,\ \ \sigma(x_i)\ =\ |x_i|^{\rho_i} \tag{5. 9}
$$

both satisfy the sufficient conditions of Stein-Weiss inequality [3], defined on $\mathbb{R}^{\mathbf{N}_i}$ for every $i = 1, 2, \dots, n$.

Let $j \in \{1,2,\ldots,n\}$ and write $x = \left(x_j, x_j^\dagger\right) \in \mathbb{R}^{\mathbf{N}_j} \times \mathbb{R}^{\mathbf{N}-\mathbf{N}_j}$. By applying Stein-Weiss inequality [3] on every subspace $\mathbb{R}^{\mathbf{N}_i}, i = 1,2,\ldots,n$, we have

$$\begin{aligned}
&\left\{\int_{|x_j|\sim|x|}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)|x|^{\eta} \prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y|}\right)^{\rho+\eta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \\
\lesssim\;& \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y) \prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y|}\right)^{\rho}\left(\frac{|x_j|}{|y_j|}\right)^{\eta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \\
\lesssim\;& \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y) \left\{\prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y_i|}\right)^{\rho_i}\right\}\left(\frac{|x_j|}{|y_j|}\right)^{\eta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \quad \text{by (5. 4)} \\
=\;& \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y) \left\{\prod_{i\neq j}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y_i|}\right)^{\rho_i}\right\}\left(\frac{1}{|x_j-y_j|}\right)^{\mathbf{N}_j-\alpha_j}\left(\frac{1}{|y_j|}\right)^{\rho_j}\left(\frac{|x_j|}{|y_j|}\right)^{\eta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \\
\lesssim\;& \left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}} f\left(x_j, y_j^\dagger\right)\prod_{i\neq j}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y_i|}\right)^{\rho_i} dy_j^\dagger\right\}^{p} dx_j\right\}^{\frac{q}{p}} dx_j^\dagger\right\}^{\frac{1}{q}} \\
\lesssim\;& \left\{\int_{\mathbb{R}^{\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_j}} f\left(x_j, y_j^\dagger\right)\prod_{i\neq j}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y_i|}\right)^{\rho_i} dy_j^\dagger\right\}^{q} dx_j^\dagger\right\}^{\frac{p}{q}} dx_j\right\}^{\frac{1}{p}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Minkowski integral inequality} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\lesssim\;& \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left(f(x)\right)^{p} dx\right\}^{\frac{1}{p}} \qquad \text{for } f \in \mathbf{L}^{p}\left(\mathbb{R}^{\mathbf{N}}\right) \text{ and } 1 < p \leq q < \infty.
\end{aligned} \tag{5. 10}$$

By applying Minkowski inequality and summing over all $j = 1,2,\ldots,n$ in (5. 7) and (5. 10), we prove the main theorem in the case of $\gamma \geq 0, \delta \leq 0$ and $\gamma \leq 0, \delta \geq 0$ respectively.

6 Reduction to Strict Subbalanced Indices

Case Three: Consider $\gamma > 0, \delta > 0$. Recall from (4. 6). We partition the set $\{1,2,\ldots n\}$ into $\mathbf{I} \cup \mathbf{J}$ such that

$$\mathbf{I} \doteq \left\{i \in \{1,2,\ldots,n\} : \frac{\alpha_i}{\mathbf{N}_i} = \frac{1}{p} - \frac{1}{q}\right\} \tag{6. 1}$$

and

$$\mathbf{J} \doteq \left\{i \in \{1,2,\ldots,n\} : \frac{\alpha_i}{\mathbf{N}_i} > \frac{1}{p} - \frac{1}{q}\right\}. \tag{6. 2}$$

We then write $x = \left(x_{\mathbf{I}}, x_{\mathbf{J}}\right) \in \mathbb{R}^{\mathbf{N_I}} \times \mathbb{R}^{\mathbf{N_J}}$ where $\mathbb{R}^{\mathbf{N_I}} \doteq \bigotimes_{i\in\mathbf{I}} \mathbb{R}^{\mathbf{N}_i}$ and $\mathbb{R}^{\mathbf{N_J}} \doteq \bigotimes_{i\in\mathbf{J}} \mathbb{R}^{\mathbf{N}_i}$.

Let $\mathbf{Q}_i$ to be centered on the origin of $\mathbb{R}^{\mathbf{N}_i}$ for every $i \in \mathbf{I}$. We define

$$\mathbf{Q_J} ~=~ \bigotimes_{i\in\mathbf{J}} \mathbf{Q}_j ~\subset~ \bigotimes_{i\in\mathbf{J}} \mathbb{R}^{\mathbf{N}_j} ~=~ \mathbb{R}^{\mathbf{N_J}}. \tag{6. 3}$$

By shrinking $\mathbf{Q}_i$ to the origin for every $i \in \mathbf{I}$ and applying Lebesgue Differentiation theorem, we have

$$\begin{aligned} \infty ~>~ & \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx\right\}^{\frac{p-1}{p}} \\ ~=~ & \prod_{i\in\mathbf{J}} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q_J}|}\int_{\mathbf{Q_J}} \left(\frac{1}{|x_{\mathbf{J}}|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q_J}|}\int_{\mathbf{Q_J}} \left(\frac{1}{|x_{\mathbf{J}}|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx\right\}^{\frac{p-1}{p}} \end{aligned} \tag{6. 4}$$

for every $\mathbf{Q_J} \in \mathbb{R}^{\mathbf{N_J}}$.

Hence that we have $\omega(x_{\mathbf{J}}) = |x_{\mathbf{J}}|^{-\gamma}, \sigma(x_{\mathbf{J}}) = |x_{\mathbf{J}}|^{\delta} \in \mathbf{A}^{\alpha}_{pq}$ on the subspace $\mathbb{R}^{\mathbf{N_J}}$. Set the hypothesis:

(**H**) *Suppose that Theorem A can be proved under the assumption of the strict subbalanced indices:* $\alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) > 0$, $i = 1, 2, \ldots, n$.

Let $f \in \mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)$ for $1 < p \le q < \infty$. We have

$$\begin{aligned} & \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y) \left(\frac{1}{|x|}\right)^{\gamma} \prod_{i=1}^{n} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} \left(\frac{1}{|y|}\right)^{\delta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \\ \lesssim~ & \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y) \left(\frac{1}{|x_{\mathbf{J}}|}\right)^{\gamma} \prod_{i=1}^{n} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} \left(\frac{1}{|y_{\mathbf{J}}|}\right)^{\delta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \qquad (\gamma > 0, \delta > 0) \\ =~ & \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y) \prod_{i\in\mathbf{I}} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} \left(\frac{1}{|x_{\mathbf{J}}|}\right)^{\gamma} \prod_{i\in\mathbf{J}} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} \left(\frac{1}{|y_{\mathbf{J}}|}\right)^{\delta} dy\right\}^{q} dx\right\}^{\frac{1}{q}} \\ \lesssim~ & \left\{\int_{\mathbb{R}^{\mathbf{N_I}}} \left\{\int_{\mathbb{R}^{\mathbf{N_J}}} \left\{\int_{\mathbb{R}^{\mathbf{N_I}}} f\left(y_{\mathbf{I}}, x_{\mathbf{J}}\right) \prod_{i\in\mathbf{I}} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} dy_{\mathbf{I}}\right\}^{p} dx_{\mathbf{J}}\right\}^{\frac{q}{p}} dx_{\mathbf{I}}\right\}^{\frac{1}{q}} \qquad \text{by } (\mathbf{H}) \\ \lesssim~ & \left\{\int_{\mathbb{R}^{\mathbf{N_J}}} \left\{\int_{\mathbb{R}^{\mathbf{N_I}}} \left\{\int_{\mathbb{R}^{\mathbf{N_I}}} f\left(y_{\mathbf{I}}, x_{\mathbf{J}}\right) \prod_{i\in\mathbf{I}} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} dy_{\mathbf{I}}\right\}^{q} dx_{\mathbf{I}}\right\}^{\frac{p}{q}} dx_{\mathbf{J}}\right\}^{\frac{1}{p}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad \text{by Minkowski integral inequality} \\ \lesssim~ & \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f(x)\right)^{p} dx\right\}^{\frac{1}{p}} \qquad \text{by the estimates in the previous section for } \gamma = \delta = 0. \end{aligned} \tag{6. 5}$$

7 Decaying Estimates and Interpolations

Let $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta} \in \mathbf{A}_{pq}^{\alpha}$ for $\gamma > 0, \delta > 0$. We prove Theorem A in the case of

$$\frac{\alpha_i}{\mathbf{N}_i} > \frac{1}{p} - \frac{1}{q}, \qquad i = 1,2,\ldots,n. \tag{7. 1}$$

Recall from (3. 6) where $|\mathbf{Q}_\imath|^{\frac{1}{\mathbf{N}_\imath}} \doteq \max_{i\in\{1,2,\ldots,n\}} |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}$. Let $0 < \lambda_i < 1$ such that

$$\lambda_i = \frac{|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}}{|\mathbf{Q}_\imath|^{\frac{1}{\mathbf{N}_\imath}}}, \qquad i = 1,2,\ldots,n. \tag{7. 2}$$

We aim to show that

$$\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q r} dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\left(\frac{pr}{p-1}\right)} dx\right\}^{\frac{p-1}{pr}} \lesssim \prod_{i=1}^{n} (\lambda_i)^{\varepsilon} \tag{7. 3}$$

for some $\varepsilon > 0, r > 1$ and every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$ satisfying (7. 2).

The implied constant in (7. 3) depends only on $p, q, \gamma, \delta, \alpha, n$ and $\mathbf{N}$. The estimate in (4. 9) shows that it is suffice to take $|\mathbf{Q}_\imath|^{\frac{1}{\mathbf{N}_\imath}} = 1$. Let $\mathbf{Q}_i^* \subset \mathbb{R}^{\mathbf{N}_i}$ to be centered on the origin and $|\mathbf{Q}_i^*|^{\frac{1}{\mathbf{N}_i}} = 3|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 3\lambda_i$ for every $i = 1,2,\ldots,n$. By taking necessary permutations, we can assume $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ without losing of the generality.

Let $0 \leq k \leq n-1$. A series of direct computations show that

$$\begin{aligned}
&\frac{1}{r}\left(\frac{1}{q} + \frac{p-1}{p}\right)\sum_{i=1}^{k} \mathbf{N}_i - (\gamma+\delta) + \sum_{i=k+1}^{n} \alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \frac{1}{r}\left(\frac{1}{q} + \frac{p-1}{p}\right)\mathbf{N} - (\gamma+\delta) + \sum_{i=k+1}^{n} \alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{r}\left(\frac{1}{q} + \frac{p-1}{p}\right)\sum_{i=k+1}^{n} \mathbf{N}_i \\
&= \frac{1}{r}\left(\frac{1}{q} + \frac{p-1}{p}\right)\mathbf{N} - (\gamma+\delta) + \sum_{i=k+1}^{n} \alpha_i - \frac{\mathbf{N}_i}{r} - \mathbf{N}_i\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \frac{\mathbf{N}}{r} - \frac{1}{r}\left(\frac{1}{p} - \frac{1}{q}\right)\mathbf{N} - (\gamma+\delta) + \sum_{i=k+1}^{n} \alpha_i - \frac{\mathbf{N}_i}{r} - \mathbf{N}_i\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \frac{\mathbf{N}}{r} - \frac{1}{r}\left(\frac{1}{p} - \frac{1}{q}\right)\mathbf{N} - \alpha + \mathbf{N}\left(\frac{1}{p} - \frac{1}{q}\right) + \sum_{i=k+1}^{n} \alpha_i - \frac{\mathbf{N}_i}{r} - \mathbf{N}_i\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) \qquad \text{by (4. 7)} \\
&= \left(\frac{\mathbf{N}}{r} - \alpha\right) + \mathbf{N}\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) + \sum_{i=k+1}^{n} \alpha_i - \frac{\mathbf{N}_i}{r} - \mathbf{N}_i\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) \\
&= \sum_{i=1}^{k} \frac{\mathbf{N}_i}{r} - \alpha_i + \mathbf{N}_i\left(1 - \frac{1}{r}\right)\left(\frac{1}{p} - \frac{1}{q}\right) \qquad \text{by (1. 2).}
\end{aligned} \tag{7. 4}$$

We momentarily consider

$$\frac{1}{q}\sum_{i=1}^{m-1}\mathbf{N}_i ~<~ \gamma ~<~ \frac{1}{q}\sum_{i=1}^{m}\mathbf{N}_i, \qquad 1\le m\le n, \tag{7. 5}$$

$$\left(\frac{p-1}{p}\right)\sum_{i=1}^{l-1}\mathbf{N}_i ~<~ \delta ~<~ \left(\frac{p-1}{p}\right)\sum_{i=1}^{l}\mathbf{N}_i, \qquad 1\le l\le n. \tag{7. 6}$$

By letting r to be sufficiently close to 1, we have

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma qr}dx\right\}^{\frac{1}{qr}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{pr}{p-1}\right)}dx\right\}^{\frac{p-1}{pr}}
\\
&\lesssim~ \prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\left(\frac{p-1}{pr}\right)}
\\
&\qquad\times~\left\{\int\cdots\int_{\bigotimes_{i=m}^n\mathbf{Q}_i}\left\{\int\cdots\int_{\bigotimes_{i=1}^{m-1}\mathbb{R}^{\mathbf{N}_i}}\left(\frac{1}{|x_1|+\cdots+|x_n|}\right)^{\gamma qr}dx_1\cdots dx_{m-1}\right\}dx_m\cdots dx_n\right\}^{\frac{1}{qr}}
\\
&\qquad\times~\left\{\int\cdots\int_{\bigotimes_{i=l}^n\mathbf{Q}_i}\left\{\int\cdots\int_{\bigotimes_{i=1}^{l-1}\mathbb{R}^{\mathbf{N}_i}}\left(\frac{1}{|x_1|+\cdots+|x_n|}\right)^{\delta\left(\frac{pr}{p-1}\right)}dx_1\cdots dx_{l-1}\right\}dx_l\cdots dx_n\right\}^{\frac{p-1}{pr}}
\\
&\lesssim~ \prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\left(\frac{p-1}{pr}\right)}
\\
&\qquad\times~\left\{\int\cdots\int_{\bigotimes_{i=m}^n\mathbf{Q}_i}\left(\frac{1}{|x_m|+\cdots+|x_n|}\right)^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i}dx_m\cdots dx_n\right\}^{\frac{1}{qr}}
\\
&\qquad\times~\left\{\int\cdots\int_{\bigotimes_{i=l}^n\mathbf{Q}_i}\left(\frac{1}{|x_l|+\cdots+|x_n|}\right)^{\delta\left(\frac{pr}{p-1}\right)-\sum_{i=1}^{l-1}\mathbf{N}_i}dx_l\cdots dx_n\right\}^{\frac{p-1}{pr}}
\\
&\lesssim~ \prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^m\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^l\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\left(\frac{p-1}{pr}\right)}
\\
&\qquad\times~\left\{\int_{\mathbf{Q}_m^*}\left(\frac{1}{|x_m|}\right)^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i}dx_m\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_l^*}\left(\frac{1}{|x_l|}\right)^{\delta\left(\frac{pr}{p-1}\right)-\sum_{i=1}^{l-1}\mathbf{N}_i}dx_l\right\}^{\frac{p-1}{pr}}
\\
&\lesssim~ (\lambda_m)^{\frac{1}{qr}\sum_{i=1}^m\mathbf{N}_i-\gamma}\,(\lambda_l)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^l\mathbf{N}_i-\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^m\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^l\left(\frac{1}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}.
\end{aligned} \tag{7. 7}$$

Notice that the implied constants among all estimates in (7. 7) depend only on dimension $\mathbf{N}$, the number of parameters n, p and q, the powers γ, δ and the fractional exponent α.

Suppose $l \leq m$. The last line of (7. 7) can be rewritten as

$$\begin{aligned}
&(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m}\mathbf{N}_i-\gamma}\,(\lambda_l)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{m}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l-1}\left(\frac{1}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}\\
&= (\lambda_m)^{\frac{1}{r}\left(\frac{1}{q}+\frac{p-1}{p}\right)\sum_{i=1}^{l-1}\mathbf{N}_i-(\gamma+\delta)}\left(\frac{\lambda_l}{\lambda_m}\right)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=l}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l-1}\left(\frac{1}{\lambda_i}\right)^{\frac{1}{r}\left(\frac{1}{q}+\frac{p-1}{p}\right)\mathbf{N}_i}\\
&= (\lambda_m)^{\frac{1}{r}\left(\frac{1}{q}+\frac{p-1}{p}\right)\sum_{i=1}^{l-1}\mathbf{N}_i+\sum_{i=l}^{n}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-(\gamma+\delta)}\left(\frac{\lambda_l}{\lambda_m}\right)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\\
&\quad\times\prod_{i=l}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=l}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l-1}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&= \left(\frac{\lambda_l}{\lambda_m}\right)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\prod_{i=l}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=l}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l-1}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i} \qquad \text{by (7. 4)}\\
&= \left(\frac{\lambda_l}{\lambda_m}\right)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=l}^{m}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\frac{\mathbf{N}_i}{p}+\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\prod_{i=1}^{l-1}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&= \left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i=1}^{l-1}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=l}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\prod_{i=1}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}.
\end{aligned} \tag{7. 8}$$

Let $\mathbf{U}$ to be defined in (4. 25) where $\alpha_i - \mathbf{N}_i/p < 0$ for every $i \notin \mathbf{U}$.

For r sufficiently close to 1, we have

$$\begin{aligned}
&\left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i=1}^{l-1}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}\prod_{i=l}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\\
&\lesssim \left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i\in\mathbf{U}\cap\{1,\ldots,l-1\}}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}\left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i\in\mathbf{U}\cap\{l,\ldots,m\}}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\\
&\quad\times\left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i\in\mathbf{U}^c\cup\{1,\ldots,l-1\}}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\prod_{i\in\mathbf{U}^c\cap\{l,\ldots,m\}}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\\
&\lesssim \left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i\in\mathbf{U}\cup\{1,\ldots,m\}}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}.
\end{aligned} \tag{7. 9}$$

Because of (7. 1), δ satisfies the strict inequality in (4. 29). For r sufficiently close to 1, we have

$$\sum_{i\in\mathbf{U}\cup\{1,\ldots,m\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta \;>\; 0. \tag{7. 10}$$

By bringing the estimate in (7. 8)-(7. 10) back to (7. 7), we have

$$\begin{aligned}
&(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m}\mathbf{N}_i-\gamma}\,(\lambda_l)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{m}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l}\left(\frac{1}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}\\
&\lesssim\;\left(\frac{\lambda_m}{\lambda_l}\right)^{\sum_{i\in\mathbf{U}\cup\{1,2,\ldots,m\}}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}\prod_{i=1}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned}\tag{7. 11}$$

From (1. 2) and (7. 1), we have $\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i<\alpha_i<\mathbf{N}_i$ for every $i=1,2,\ldots,n$. Observe that (7. 11) is bounded by a constant multiple of $(\lambda_n)^{\varepsilon}$ for some $\varepsilon>0$ and every $\lambda_l\geq\lambda_m\geq\lambda_n$ provided that r is sufficiently close to 1.

Suppose $m\leq l$. The last line of (7. 7) can be rewritten as

$$\begin{aligned}
&(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\,(\lambda_l)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{m-1}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l}\left(\frac{1}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}\\
&=\;(\lambda_l)^{\frac{1}{r}\left(\frac{1}{q}+\frac{p-1}{p}\right)\sum_{i=1}^{m-1}\mathbf{N}_i-(\gamma+\delta)}\left(\frac{\lambda_m}{\lambda_l}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{1}{\lambda_i}\right)^{\frac{1}{r}\left(\frac{1}{q}+\frac{p-1}{p}\right)\mathbf{N}_i}\\
&=\;(\lambda_l)^{\frac{1}{r}\left(\frac{1}{q}+\frac{p-1}{p}\right)\sum_{i=1}^{m-1}\mathbf{N}_i+\sum_{i=m}^{n}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-(\gamma+\delta)}\left(\frac{\lambda_m}{\lambda_l}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\\
&\qquad\times\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_l}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&=\;\left(\frac{\lambda_m}{\lambda_l}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_l}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{\lambda_l}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\qquad\text{by (7. 4)}\\
&=\;\left(\frac{\lambda_m}{\lambda_l}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\prod_{i=l}^{n}\left(\frac{\lambda_i}{\lambda_l}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{l}\left(\frac{\lambda_i}{\lambda_l}\right)^{\alpha_i-\left(\frac{q-1}{q}\right)\mathbf{N}_i+\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{\lambda_l}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&=\;\left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i=1}^{m-1}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma}\prod_{i=l}^{n}\left(\frac{\lambda_i}{\lambda_l}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\\
&\qquad\times\prod_{i=m}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i}\prod_{i=1}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}.
\end{aligned}\tag{7. 12}$$

Let $\mathbf{V}$ to be defined in (4. 25) where $\alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) < 0$ for every $i \notin \mathbf{V}$.

For r sufficiently close to 1, we have

$$\begin{aligned}
&\left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i=1}^{m-1}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma} \prod_{i=m}^{l}\left(\frac{\lambda_l}{\lambda_i}\right)^{\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i} \\
\lesssim\;& \left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i\in\mathbf{V}\cap\{1,\ldots,m-1\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma} \left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i\in\mathbf{V}\cap\{m,\ldots,l\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i} \\
&\times \left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i\in\mathbf{V}^c\cap\{1,\ldots,m-1\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i} \prod_{i\in\mathbf{V}^c\cap\{m,\ldots,l\}}\left(\frac{\lambda_l}{\lambda_i}\right)^{\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i} \\
\lesssim\;& \left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i\in\mathbf{V}\cap\{1,\ldots,l\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma}.
\end{aligned} \tag{7. 13}$$

Because of (7. 1), γ satisfies the strict inequality in (4. 33). For r sufficiently close to 1, we have

$$\sum_{i\in\mathbf{V}\cap\{1,\ldots,l\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma \;>\; 0. \tag{7. 14}$$

By bringing the estimate in (7. 12)-(7. 14) back to (7. 7), we have

$$\begin{aligned}
&(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m}\mathbf{N}_i-\gamma}\,(\lambda_l)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{l}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{m}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{l}\left(\frac{1}{\lambda_i}\right)^{\left(\frac{p-1}{pr}\right)\mathbf{N}_i} \\
\lesssim\;& \left(\frac{\lambda_l}{\lambda_m}\right)^{\sum_{i\in\mathbf{V}\cap\{1,\ldots,l\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma} \\
&\times \prod_{i=1}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\prod_{i=l}^{n}\left(\frac{\lambda_i}{\lambda_l}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned} \tag{7. 15}$$

From (1. 2) and (7. 1), we have $\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i < \alpha_i < \mathbf{N}_i$ for every $i = 1, 2, \ldots, n$. Observe that (7. 15) is bounded by a constant multiple of $(\lambda_n)^{\varepsilon}$ for some $\varepsilon > 0$ and every $\lambda_m \geq \lambda_l \geq \lambda_n$ provided that r is sufficiently close to 1.

Notice that we have $(\lambda_n)^{\varepsilon} \leq \prod_{i=1}^{n}(\lambda_i)^{\frac{\varepsilon}{n}}$ for $\varepsilon > 0$ since $1 = \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$. From the estimates in (7. 4)-(7. 15), we prove the result in (7. 2)-(7. 3). Recall the definition of $\mathbf{A}^{\alpha}_{pqr}(\mathbf{t} : \omega, \sigma)$ in (3. 6)-(3. 7). Let t_i to be an nonnegative integer such that $2^{-t_i-1} \leq \lambda_i \leq 2^{-t_i+1}$ for every $i = 1, 2, \ldots, n$. We have

$$\mathbf{A}^{\alpha}_{pqr}(\mathbf{t} : \omega, \sigma) \lesssim \prod_{i=1}^{n} 2^{-\varepsilon t_i}$$

for some $\varepsilon > 0$ and every given $\mathbf{t}$, provided that r is sufficiently close to 1 and γ, δ satisfying (7. 5)-(7. 6) respectively. This indeed implies the summability in (3. 14).

Suppose that at least one of the equalities

$$\gamma = \frac{1}{q}\sum_{i=1}^{m}\mathbf{N}_i, \qquad \delta = \left(\frac{p-1}{p}\right)\sum_{i=1}^{l}\mathbf{N}_i \tag{7. 16}$$

holds for $1 \le m, l \le n-1$.

Let $\omega_i(x) = |x|^{-\gamma_i}, \sigma_i(x) = |x|^{\delta_i}$ for $\gamma_i, \delta_i \in \mathbb{R}$ and $i = 1, 2$. We define

$$\gamma_i = \gamma \pm \varepsilon, \qquad \delta_i = \delta \mp \varepsilon, \qquad i = 1, 2 \tag{7. 17}$$

for some $\varepsilon > 0$ sufficiently small, satisfying the strict inequalities in (7. 5)-(7. 6) respectively.

Observe that γ_1, δ_1 and γ_2, δ_2 defined in (7. 17) both satisfy the formula in (4. 7) for given $p, q, \alpha, \mathbf{N}$. On the other hand, because of (7. 1), γ, δ satisfy the strict inequalities in (4. 33) and (4. 29) respectively. These inequalities remain to be true for γ_1, δ_1 and γ_2, δ_2, provided that ε in (7. 17) is sufficiently close to 0. By carrying out the same estimates from (7. 4) to (7. 15) with γ, δ replaced by γ_1, δ_1 and γ_2, δ_2, we obtain the result in (7. 2)-(7. 3) for $\omega_1(x) = |x|^{-\gamma_1}, \sigma_1(x) = |x|^{\delta_1}$ and $\omega_2(x) = |x|^{-\gamma_2}, \sigma_2(x) = |x|^{\delta_2}$. The argument below (7. 15) thus implies simultaneously for $\omega_1 \mathbf{I}_\alpha \sigma_1^{-1}\colon \mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right) \longrightarrow \mathbf{L}^q\left(\mathbb{R}^{\mathbf{N}}\right)$ and $\omega_2 \mathbf{I}_\alpha \sigma_2^{-1}\colon \mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right) \longrightarrow \mathbf{L}^q\left(\mathbb{R}^{\mathbf{N}}\right)$. By applying Stein interpolation theorem of linear operators, stated as Theorem 2 in Stein [4], we prove the desired result for $\gamma > 0, \delta > 0$.

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