Singular Integrals of Nonconvolution Type on Product Spaces

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Abstract

A new class of symbols is investigated. These symbols satisfy a differential inequality which has a mixture of homogeneities on the product space. We show that pseudo differential operators with symbols in this class are $\mathbf{L}^p$-bounded for $1 < p < \infty$. Moreover, they form an algebra under compositions.

1 Introduction

The $\mathbf{L}^p$-regularity of singular integrals defined on the product space has been studied by Robert Fefferman and Stein [1] and later by Nagel, Ricci and Stein [2] for some convolution operators, commuting with a multi-parameters family of dilations. Their kernels essentially satisfy the Calderón-Zygmund condition on each of the coordinate subspaces. In this paper, we generalize the result into a variable coefficient setting, by studying singular integrals with nonconvolution kernels satisfying the desired size estimate and cancellation property. On the other hand, they can be characterized as a class of pseudo differential operators whose symbols satisfy certain characteristic property on the product space. We prove the $\mathbf{L}^p$-boundedness of these operators for $1 < p < \infty$ and show that they form an algebra under compositions.

Let $\mathbb{R}^{\mathbf{N}} = \mathbb{R}^{\mathbf{N}_1} \times \mathbb{R}^{\mathbf{N}_2} \times \cdots \times \mathbb{R}^{\mathbf{N}_n}$ and define the inner product

$$x \cdot \xi \;=\; x_1 \cdot \xi_1 + x_2 \cdot \xi_2 + \cdots + x_n \cdot \xi_n \tag{1. 1}$$

where ξ_i is the dual variable of $x_i \in \mathbb{R}^{\mathbf{N}_i}$ for every $i = 1, 2, \ldots, n$.

Let $f \in \mathcal{S}$ be a Schwartz function. A pseudo differential operator T_σ is defined by

$$\Big(T_\sigma f\Big)(x) \;=\; \int_{\mathbb{R}^{\mathbf{N}}} \sigma(x,\xi) e^{2\pi \mathbf{i} x\cdot\xi} \widehat{f}(\xi) d\xi \tag{1. 2}$$

with symbol $\sigma(x,\xi) \in C^\infty\Big(\mathbb{R}^{\mathbf{N}} \times \mathbb{R}^{\mathbf{N}}\Big)$.

We always write A as a positive, generic constant with subindices indicating its dependences. Our class of pseudo differential operators is defined by classifying $\sigma(x,\xi)$ in the following symbol class.

Symbol Class $\mathbf{S}_\rho$: Let $0 \leq \rho < 1$. A symbol $\sigma \in \mathbf{S}_\rho$ if it satisfies

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{\partial}{\partial x}\right)^\beta \sigma(x,\xi)\right| \leq A_{\alpha\beta} \prod_{i=1}^n \left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha_i|} (1+|\xi|)^{\rho|\beta|} \tag{1. 3}$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and β.

For $\rho = 0$, the symbol $\sigma(x, \cdot)$ belonging to $\mathbf{S}_\rho$ is essentially a *product symbol*, as was first investigated by Nagel, Ricci and Stein [2]. In general, $\mathbf{S}_\rho$ forms a proper subclass of the *exotic* symbol class $S^0_{\rho,\rho}$. It is well known that the Fourier transform of *Riemann singularity* belongs to $S^0_{\frac{1}{2},\frac{1}{2}}$ and the corresponding Fourier multiplier operator is bounded only on the $\mathbf{L}^2$-spaces. See chapter VII of the book by Stein [7]. Apart from that, $\sigma \in \mathbf{S}_\rho$ satisfies a variant of the Marcinkiewicz condition [3]:

$$\left|\left(\xi\frac{\partial}{\partial \xi}\right)^\alpha \sigma(x,\xi)\right| \leq A_\alpha \tag{1. 4}$$

for every multi-index α, uniformly in x. By applying Marcinkiewicz multiplier theorem [6], if $\sigma = \sigma(\xi) \in \mathbf{S}_\rho$, then T_σ is bounded on $\mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)$ for $1 < p < \infty$.

On the other hand, by taking singular integral realization, the kernel of T_σ defined in (1. 2) coincides with a smooth function

$$\Omega(x,y) \doteq \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i}(x-y)\cdot\xi} \sigma(x,\xi) d\xi \tag{1. 5}$$

away from its singularity.

Let $z = x - y$ and omit the notations $\Omega(x,y) = \Omega(x,z)$. Recall that an normalized bump function is smooth and equals 1 near the origin, supported on the unit ball with all its derivatives bounded upon to a sufficiently large order. Consider $\mathbf{I} \cup \mathbf{J} = \{1, 2, \dots, n\}$. Let φ_i to be an normalized bump function for $i \in \mathbf{I}$ and $z_\mathbf{J}$ is the projection of z on the subspace $\bigoplus_{j\in\mathbf{J}} \mathbb{R}^{\mathbf{N}_j}$. The following estimates hold, if and only if, $\sigma \in \mathbf{S}_\rho$.

Size Estimate: For every multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and β, Ω in (1. 5) satisfies

$$\left|\left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial x}\right)^\beta \Omega(x,z)\right| \leq A_{\alpha\beta} \prod_{i=1}^n \left(\frac{1}{|z_i|+|z|^{1/\rho}}\right)^{\mathbf{N}_i+|\alpha^i|+\rho|\beta|} \tag{1. 6}$$

at $z \neq 0$ when $0 < \rho < 1$ and $z_i \neq 0, i = 1, 2, \dots, n$ when $\rho = 0$, and decays rapidly as $|z| \longrightarrow \infty$.

Cancellation Property: For every multi-index α_j, $j \in \mathbf{J}$ and β, Ω in (1. 5) satisfies

$$\begin{aligned} &\left|\prod_{j\in\mathbf{J}}\left(\frac{\partial}{\partial z_j}\right)^{\alpha_j} \int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}} \left(\left(\frac{\partial}{\partial x}\right)^\beta \Omega(x,z)\right) \prod_{i\in\mathbf{I}} \varphi_i(R_i z_i) dz_i\right| \\ &\leq A_{\alpha\beta} \prod_{j\in\mathbf{J}} \left(\frac{1}{|z_j|+|z_\mathbf{J}|^{1/\rho}}\right)^{\mathbf{N}_j+|\alpha_j|+\rho|\beta|} \left(1+\sum_{i\in\mathbf{I}} R_i\right)^{\rho|\beta|} \end{aligned} \tag{1. 7}$$

for every $R_i > 0, i \in \mathbf{I}$, at $z_\mathbf{J} \neq 0$ when $0 < \rho < 1$ and $z_j \neq 0, j \in \mathbf{J}$ when $\rho = 0$, and decays rapidly as $|z_\mathbf{J}| \longrightarrow \infty$.

Observe that when $\rho = 0$, the kernel of T_σ for $\sigma \in \mathbf{S}_\rho$ has singularity appeared on each of the coordinate subspaces. It is essentially a *product kernel* introduced in the first section of [2]. Our main result is stated in below.

Main Theorem: *Let $\sigma \in \mathbf{S}_\rho$. Pseudo differential operator T_σ in (1. 2), initially defined on $\mathcal{S}$, extends to a bounded operator on $\mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)$ for $1 < p < \infty$.*

Sketch of Proof: In section 2, we prove a principal lemma. As a corollary, we show that T_σ with $\sigma \in \mathbf{S}_\rho$ form an algebra. In section 3, we introduce an appropriate Littlewood-Paley projections and give certain combinatorial estimates on the regarding Dyadic decomposition. In section 4, we give a classification for $\sigma \in \mathbf{S}_\rho$ and Ω satisfying (1. 6)-(1. 7). In section 5, we show that every partial sum operator in the framework is bounded by the strong maximal function. In section 6, we further prove a decaying estimate on these partial sum operators. We conclude the main theorem in section 7. We add an appendix in the end for the required Littlewood-Paley inequality.

Abbreviations:

◇ Unless otherwise indicated, we write $\int = \int_{\mathbb{R}^{\mathbf{N}}}$, $\iint = \iint_{\mathbb{R}^{\mathbf{N}}\times\mathbb{R}^{\mathbf{N}}}$ and $\mathbf{L}^p = \mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)$.

2 A Principal Lemma

Our analysis in this section shares the same sprit of the work by Boutet de Monvel [8] also Beals and C. Fefferman [9].

Let $\varphi = \varphi(x, y, \xi, \eta)$ to be a smooth function with norm bounded by the norm function $\vartheta(\xi, \eta)$. Let $0 \leq \rho < 1$. We consider

$$\begin{array}{c}\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha^1}\left(\frac{\partial}{\partial\eta}\right)^{\alpha^2}\left(\frac{\partial}{\partial x}\right)^{\beta^1}\left(\frac{\partial}{\partial y}\right)^{\beta^2}\varphi(x, y, \xi, \eta)\right| \\ \leq A_{\alpha^1\alpha^2\beta^1\beta^2}\vartheta(\xi, \eta)\prod_{i=1}^{n}\left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha_i^1|}\left(\frac{1}{1+|\eta_i|+|\eta|^\rho}\right)^{|\alpha_i^2|}(1+|\xi|)^{\rho|\beta^1|}(1+|\eta|)^{\rho|\beta^2|}\end{array} \tag{2. 1}$$

for every multi-index $\alpha^1 = \left(\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1\right)$, $\alpha^2 = \left(\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2\right)$ and β^1, β^2.

Define

$$\Lambda(x, \xi) = \iint e^{2\pi\mathbf{i}(x-y)\cdot(\xi-\eta)}\varphi(x, y, \xi, \eta)dyd\eta. \tag{2. 2}$$

The main objective of the section is to prove the following:

Lemma 2.1 *Suppose that $\varphi(x, y, \xi, \eta)$ is bounded and satisfies the differential inequality in (2. 1). Then, $\Lambda(x, \xi)$ defined in (2. 2) satisfies*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}\Lambda(x, \xi)\right| \leq A_{\alpha\beta\rho}\prod_{i=1}^{n}\left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha^i|}(1+|\xi|)^{\rho|\beta|} \tag{2. 3}$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and β.

Corollary 2.1 *Let* $\sigma_1 \circ \sigma_2$ *to be the symbol of which* $T_{\sigma_1 \circ \sigma_2} = T_{\sigma_1} \circ T_{\sigma_2}$. *Suppose that* $\sigma_1, \sigma_2 \in \mathbf{S}^0_\rho$. *Then,* $\sigma_1 \circ \sigma_2 \in \mathbf{S}^0_\rho$.

Proof: A direct computation shows

$$\Big(\sigma_1 \circ \sigma_2\Big)(x, \xi) \ = \ \iint e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)} \sigma_1(x, \eta)\sigma_2(y, \xi) dy d\eta.$$

The function $\sigma_1(x, \eta)\sigma_2(y, \xi)$ satisfies the differential inequality in (2. 1). By Lemma 2.1, the symbol $\sigma_1 \circ \sigma_2$ satisfies the differential inequality in (1. 3) for $m = 0$. □

Remark 2.1 *We will momentarily assume that* φ *has a compact support in* y. *But, our estimates are independent from its size.*

Let $0 \le \rho < 1$. Define the differential operator

$$D \ = \ I - \left(\frac{1}{4\pi^2}\right)\left(\frac{1}{1+|\xi|^2} + \frac{1}{1+|\eta|^2}\right)^{\rho} \Delta_y - \left(\frac{1}{4\pi^2}\right)\Big(1 + |\xi|^2 + |\eta|^2\Big)^{\rho} \Delta_\eta \tag{2. 4}$$

and respectively the quadratic function

$$Q(x, y, \xi, \eta) \ = \ 1 + \left(\frac{1}{1+|\xi|^2} + \frac{1}{1+|\eta|^2}\right)^{\rho} |\xi - \eta|^2 + \Big(1 + |\xi|^2 + |\eta|^2\Big)^{\rho} |x - y|^2. \tag{2. 5}$$

Let $L = Q^{-1}D$. We have $L^N\Big(e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)}\Big) \ = \ e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)}$ for every $N \ge 1$. Integration by parts with respect to y and η inside (2. 2) gives

$$\iint \left({}^tL\right)^N \Big(\varphi(x, y, \xi, \eta)\Big) e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)} dy d\eta \tag{2. 6}$$

where ${}^tL = {}^tDQ^{-1}$.

Lemma 2.2 *Let* φ *satisfying the differential inequality in (2. 1). We have*

$$\left|\left({}^tL\right)^N \varphi(x, y, \xi, \eta)\right| \ \le \ A_N \ \vartheta(\xi, \eta) \left(1 + \frac{|\xi|^2}{1+|\eta|^2} + \frac{|\eta|^2}{1+|\xi|^2}\right)^{\rho N} \left(\frac{1}{Q}\right)^N (x, y, \xi, \eta) \tag{2. 7}$$

for every $N \ge 1$.

Proof : Let

$$a(\xi, \eta) \ = \ \left(\frac{1}{1+|\xi|^2} + \frac{1}{1+|\eta|^2}\right)^{\rho}, \qquad b(\xi, \eta) \ = \ \Big(1 + |\xi|^2 + |\eta|^2\Big)^{\rho}. \tag{2. 8}$$

From (2. 4)-(2. 5), we have

$$\left({}^tL\right)^N \varphi(x, y, \xi, \eta) \ = \ \left(\frac{1}{Q} - \Delta_y \frac{a}{Q} - \Delta_\eta \frac{b}{Q}\right)^N \varphi(x, y, \xi, \eta) \tag{2. 9}$$

for every $N \ge 1$.

Observe that

$$\begin{aligned} a(\xi,\eta)\Big|\big(\Delta_y\varphi\big)(x,y,\xi,\eta)\Big| &\lesssim a(\xi,\eta)\,(1+|\eta|)^{2\rho}\,\vartheta(\xi,\eta) \\ &\lesssim \left(1+\frac{|\xi|^2}{1+|\eta|^2}+\frac{|\eta|^2}{1+|\xi|^2}\right)^{\rho}\vartheta(\xi,\eta) \end{aligned} \tag{2. 10}$$

and

$$\begin{aligned} b(\xi,\eta)\Big|\big(\Delta_\eta\varphi\big)(x,y,\xi,\eta)\Big| &\lesssim b(\xi,\eta)\left(\frac{1}{1+|\eta|}\right)^{2\rho}\vartheta(\xi,\eta) \\ &\lesssim \left(1+\frac{|\xi|^2}{1+|\eta|^2}+\frac{|\eta|^2}{1+|\xi|^2}\right)^{\rho}\vartheta(\xi,\eta). \end{aligned} \tag{2. 11}$$

If all differentiations fall on φ in (2. 9), then (2. 10)-(2. 11) would imply the differential inequality in (2. 7).

Turn to the general case. We aim to show

$$\left|\left(\frac{\partial}{\partial\eta}\right)^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta}\left(\frac{1}{Q}\right)(x,y,\xi,\eta)\right| \lesssim \left(\frac{1}{Q}\right)(x,y,\xi,\eta)\left(\frac{1}{1+|\eta|}\right)^{\rho|\alpha|}(1+|\xi|+|\eta|)^{\rho|\beta|} \tag{2. 12}$$

for every multi-index α and β.

From (2. 5) and (2. 8), $Q(x,y,\xi,\eta)=1+a(\xi,\eta)|\xi-\eta|^2+b(\xi,\eta)|x-y|^2$. It is easy to verify that

$$\Big|\big(\partial_\eta a\big)^{\alpha}(\xi,\eta)\Big| \lesssim \left(\frac{1}{1+|\eta|}\right)^{|\alpha|}a(\xi,\eta), \qquad \Big|\big(\partial_\eta b\big)^{\alpha}(\xi,\eta)\Big| \lesssim \left(\frac{1}{1+|\xi|+|\eta|}\right)^{|\alpha|}b(\xi,\eta) \tag{2. 13}$$

for every multi-index α.

The differential inequality in (2. 7) can be obtained by applying the estimates (2. 12)-(2. 13) inside (2. 9), together with φ satisfying the differential inequality in (2. 1). We next develop a series of preliminary results:

Suppose $|\xi-\eta|\le(1+|\xi|+|\eta|)^{\rho}$. Since $0\le\rho<1$, we necessarily have $|\xi|\sim|\eta|$ and

$$\begin{aligned} \left(\frac{1}{1+|\xi|^2}+\frac{1}{1+|\eta|^2}\right)^{\rho}|\xi-\eta| &\lesssim \left(\frac{1}{1+|\eta|}\right)^{\rho}, \\ \left(\frac{1}{1+|\xi|^2}+\frac{1}{1+|\eta|^2}\right)^{\rho} &\lesssim \left(\frac{1}{1+|\eta|}\right)^{2\rho}. \end{aligned} \tag{2. 14}$$

Suppose $|\xi-\eta|>(1+|\xi|+|\eta|)^{\rho}$. We have

$$\begin{aligned} \left(\frac{1}{Q}\right)(x,y,\xi,\eta)\left(\frac{1}{1+|\xi|^2}+\frac{1}{1+|\eta|^2}\right)^{\rho}|\xi-\eta| &\lesssim \left(\frac{1}{|\xi-\eta|}\right) \lesssim \left(\frac{1}{1+|\eta|}\right)^{\rho}, \\ \left(\frac{1}{Q}\right)(x,y,\xi,\eta)\left(\frac{1}{1+|\xi|^2}+\frac{1}{1+|\eta|^2}\right)^{\rho} &\lesssim \left(\frac{1}{|\xi-\eta|^2}\right) \lesssim \left(\frac{1}{1+|\eta|}\right)^{2\rho}. \end{aligned} \tag{2. 15}$$

Suppose $|x-y| \le (1+|\xi|+|\eta|)^{-\rho}$. We have

$$\begin{aligned} \left(1+|\xi|^2+|\eta|^2\right)^{\rho} |x-y| &\lesssim (1+|\xi|+|\eta|)^{\rho}, \\ \left(1+|\xi|^2+|\eta|^2\right)^{\rho} &\lesssim (1+|\xi|+|\eta|)^{2\rho}. \end{aligned} \tag{2. 16}$$

Suppose $|x-y| > (1+|\xi|+|\eta|)^{-\rho}$. We have

$$\begin{aligned} \left(\frac{1}{Q}\right)(x,y,\xi,\eta)\left(1+|\xi|^2+|\eta|^2\right)^{\rho}|x-y| &\lesssim \left(\frac{1}{|x-y|}\right) \lesssim (1+|\xi|+|\eta|)^{\rho}, \\ \left(\frac{1}{Q}\right)(x,y,\xi,\eta)\left(1+|\xi|^2+|\eta|^2\right)^{\rho} &\lesssim \left(\frac{1}{|x-y|^2}\right) \lesssim (1+|\xi|+|\eta|)^{2\rho}. \end{aligned} \tag{2. 17}$$

Notice that for $|x-y|$ and $|\xi-\eta|$ small, we have $Q(x,y,\xi,\eta) \ge 1$ from (2. 5).

In order to prove (2. 12), observe that $\partial_\eta^\alpha \partial_y^\beta Q^{-1}$ consists a linear combination of

$$Q^{-1}\prod_j \left(Q^{-1}\partial_\eta^{\alpha^j}\partial_y^{\beta^j} Q\right)^j \tag{2. 18}$$

where $|\alpha| = \sum_j |\alpha^j| j$ and $|\beta| = \sum_j |\beta^j| j$. In particular, we have $\partial_\eta Q^{-1} = (-1)Q^{-1}\left(Q^{-1}\partial_\eta Q\right)$ and $\partial_y Q^{-1} = (-1)Q^{-1}\left(Q^{-1}\partial_y Q\right)$. By writing out the expansion for $\partial_\eta^\alpha \partial_y^\beta Q$, and using the estimates in (2. 14)-(2. 15) and (2. 16)-(2. 17) respectively, together with (2. 13), we have

$$\left|\left(\frac{1}{Q}\right)\left(\frac{\partial}{\partial\eta}\right)^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta} Q(x,y,\xi,\eta)\right| \lesssim \left(\frac{1}{1+|\eta|}\right)^{\rho|\alpha|}(1+|\xi|+|\eta|)^{\rho|\beta|} \tag{2. 19}$$

for every multi-index α and β. □

Proof of Lemma 2.1: 1. We consider $(\partial_\xi)^\alpha (\partial_x)^\beta \Lambda(x,\xi)$ consisting a linear combination of

$$\iint e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)}\left\{\left(\frac{\partial}{\partial\xi}\right)^{\alpha^1}\left(\frac{\partial}{\partial x}\right)^{\beta^1}\left(\frac{\partial}{\partial\eta}\right)^{\alpha^2}\left(\frac{\partial}{\partial y}\right)^{\beta^2}\varphi(x,y,\xi,\eta)\right\} dy d\eta \tag{2. 20}$$

where $\alpha_i = \alpha_i^1 + \alpha_i^2$ and $\beta = \beta_i^1 + \beta_i^2$ for every $i = 1, 2, \ldots, n$.

2. When $|\xi-\eta| \le \frac{1}{2}(1+|\xi|+|\eta|)$, we necessarily have $|\xi| \sim |\eta|$. The function $\partial_\xi^{\alpha^1}\partial_\eta^{\alpha^2}\partial_x^{\beta^1}\partial_y^{\beta^2}\varphi$ satisfies the differential inequality in (2. 1), with its norm bounded by

$$A_{\alpha\beta}\prod_{i=1}^n \left(\frac{1}{1+|\xi_i|+|\xi|^{\rho}}\right)^{|\alpha_i|}(1+|\xi|)^{\rho|\beta|}. \tag{2. 21}$$

Recall ${}^tL = {}^tDQ^{-1}$ as (2. 4)-(2. 5) after integration by parts with respect to y and η in (2. 20).

By applying Lemma 2.2, we have

$$\iint_{|\xi-\eta| \leq \frac{1}{2}(1+|\xi|+|\eta|)} e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)} \left\{ \left(\frac{\partial}{\partial \xi}\right)^{\alpha^1} \left(\frac{\partial}{\partial x}\right)^{\beta^1} \left(\frac{\partial}{\partial \eta}\right)^{\alpha^2} \left(\frac{\partial}{\partial y}\right)^{\beta^2} \varphi(x,y,\xi,\eta) \right\} dy d\eta$$

$$= \iint_{|\xi-\eta| \leq \frac{1}{2}(1+|\xi|+|\eta|)} e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)} \left({}^tL\right)^N \left\{ \left(\frac{\partial}{\partial \xi}\right)^{\alpha^1} \left(\frac{\partial}{\partial x}\right)^{\beta^1} \left(\frac{\partial}{\partial \eta}\right)^{\alpha^2} \left(\frac{\partial}{\partial y}\right)^{\beta^2} \varphi(x,y,\xi,\eta) \right\} dy d\eta$$

$$\leq A_{\alpha\beta N} \left\{ \iint Q^{-N}(x,y,\xi,\eta) dy d\eta \right\} \prod_{i=1}^{n} \left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha_i|} (1+|\xi|)^{\rho|\beta|}, \qquad N \geq 1. \tag{2. 22}$$

3. Suppose that $|\xi-\eta| > \frac{1}{2}(1+|\xi|+|\eta|)$. We proceed an M-fold integration by parts with respect to y in (2. 20). The resulting function $\Delta_y^M \left(\partial_\xi^{\alpha^1} \partial_\eta^{\alpha^2} \partial_x^{\beta^1} \partial_y^{\beta^2} \varphi\right) / |\xi-\eta|^{2M}$ satisfies the differential inequality in (2. 1) with norm bounded by a constant $A_{\alpha\beta M}$ multiple of

$$(1+|\xi|+|\eta|)^{2(\rho-1)M} \prod_{i=1}^{n} \left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha_i^1|} \left(\frac{1}{1+|\eta_i|+|\eta|^\rho}\right)^{|\alpha_i^2|} (1+|\xi|)^{\rho|\beta^1|} (1+|\eta|)^{\rho|\beta^2|}. \tag{2. 23}$$

By applying Lemma 2.2, we have

$$\left| \left({}^tL\right)^N \left\{ \left(\frac{1}{|\xi-\eta|}\right)^{2M} \Delta_y^M \left(\partial_\xi^{\alpha^1} \partial_\eta^{\alpha^2} \partial_x^{\beta^1} \partial_y^{\beta^2} \varphi\right) \right\} (x,y,\xi,\eta) \right| \lesssim \left(1 + \frac{|\xi|^2}{1+|\eta|^2} + \frac{|\eta|^2}{1+|\xi|^2}\right)^{\rho N} \left(\frac{1}{Q}\right)^N (x,y,\xi,\eta)$$

$$\times (1+|\xi|+|\eta|)^{2(\rho-1)M} \prod_{i=1}^{n} \left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha_i^1|} \left(\frac{1}{1+|\eta_i|+|\eta|^\rho}\right)^{|\alpha_i^2|} (1+|\xi|)^{\rho|\beta^1|} (1+|\eta|)^{\rho|\beta^2|} \tag{2. 24}$$

where the implied constant depends on α, β, M and N.

By letting M to be sufficiently large, depending on α, β, ρ and N, we have

$$\iint_{|\xi-\eta| > \frac{1}{2}(1+|\xi|+|\eta|)} e^{2\pi \mathbf{i}(x-y)\cdot(\xi-\eta)} \left({}^tL\right)^N \left\{ \left(\frac{1}{|\xi-\eta|}\right)^{2M} \Delta_y^M \left(\frac{\partial}{\partial \xi}\right)^{\alpha^1} \left(\frac{\partial}{\partial x}\right)^{\beta^1} \left(\frac{\partial}{\partial \eta}\right)^{\alpha^2} \left(\frac{\partial}{\partial y}\right)^{\beta^2} \varphi \right\} (x,y,\xi,\eta) dy d\eta$$

$$\leq A_{\alpha\beta\rho N} \left\{ \iint Q^{-N}(x,y,\xi,\eta) dy d\eta \right\} \prod_{i=1}^{n} \left(\frac{1}{1+|\xi_i|+|\xi|^\rho}\right)^{|\alpha_i|} (1+|\xi|)^{\rho|\beta|}. \tag{2. 25}$$

4. Our estimates above are independent from the size of φ's support in y. Its compactness can be removed by taking the approximation as discussed in **1.3**, chapterVI of [7].

From (2. 5), we have

$$Q(x,y,\xi,\eta) \geq 1 + \left(\frac{1}{1+|\xi|^2}\right)^\rho |\xi-\eta|^2 + \left(1+|\xi|^2\right)^\rho |x-y|^2. \tag{2. 26}$$

By changing variables

$$\eta \longrightarrow \left(1+|\xi|^2\right)^{\rho/2}(\xi-\eta) \qquad \text{and} \qquad y \longrightarrow \left(\frac{1}{1+|\xi|^2}\right)^{\rho/2}(x-y), \tag{2. 27}$$

we have

$$\iint Q^{-N}(x,y,\xi,\eta)dyd\eta \lesssim \iint \frac{dyd\eta}{\left(1+|\eta|^2+|y|^2\right)^N}. \tag{2. 28}$$

The integral converges provided that N is sufficiently large. □

3 Combinatorial Estimates

Let t_i be nonnegative integers for every $i = 1, 2, \ldots, n$. We write $\mathfrak{q} = 1/\rho$ for which $0 \leq \rho < 1$. Consider the n-tuples

$$\mathbf{t}_i = \left(2^{-\mathfrak{q}t_i},\ \ldots,\ 2^{-\mathfrak{q}t_i},\ 2^{-t_i},\ 2^{-\mathfrak{q}t_i},\ \ldots,\ 2^{-\mathfrak{q}t_i}\right) \tag{3. 1}$$

where 2^{-t_i} is located on the i-th component. We define the nonisotropic dilations

$$\mathbf{t}_i\xi = \left(2^{-\mathfrak{q}t_i}\xi_1,\ \ldots,\ 2^{-\mathfrak{q}t_i}\xi_{i-1},\ 2^{-t_i}\xi_i,\ 2^{-\mathfrak{q}t_i}\xi_{i+1},\ \ldots,\ 2^{-\mathfrak{q}t_i}\xi_n\right) \tag{3. 2}$$

for every $i = 1, 2, \ldots, n$. On the other hand, we write simultaneously

$$\mathbf{t} = \left(2^{-t_1},\ 2^{-t_2}, \ldots,\ 2^{-t_n}\right), \qquad \mathbf{t}^{-1} = \left(2^{t_1},\ 2^{t_2}, \ldots,\ 2^{t_n}\right), \tag{3. 3}$$

$$\mathbf{t}\xi = \left(2^{-t_1}\xi_1,\ 2^{-t_2}\xi_2, \ldots,\ 2^{-t_n}\xi_n\right), \qquad \mathbf{t}^{-1}\xi = \left(2^{t_1}\xi_1,\ 2^{t_2}\xi_2, \ldots,\ 2^{t_n}\xi_n\right). \tag{3. 4}$$

Let $\varphi \in C_o^\infty(\mathbb{R}^{\mathbf{N}})$ such that $\varphi \equiv 1$ for $|\xi| \leq 1$ and $\varphi = 0$ for $|\xi| \geq 2$. We define

$$\phi(\xi) = \varphi(\xi) - \varphi(2\xi) \tag{3. 5}$$

and

$$\delta_{\mathbf{t}}(\xi) = \prod_{i=1}^n \phi\left(\mathbf{t}_i\xi\right). \tag{3. 6}$$

Its support lies inside the intersection of n elliptical shells, with different homogeneities of given dilations. In particular, at $\rho = 0$ the support of $\delta_{\mathbf{t}}(\xi)$ lies inside the Dyadic rectangle $|\xi_i| \sim 2^{t_i}$, $i = 1, 2, \ldots, n$.

We define the partial sum operator $\Delta_{\mathbf{t}}$ by

$$\begin{aligned}\left(\widehat{\Delta_{\mathbf{t}}f}\right)(\xi) &= \delta_{\mathbf{t}}(\xi)\widehat{f}(\xi)\\ &= \prod_{i=1}^n \phi\left(2^{-\mathfrak{q}t_i}\xi_1, \ldots, 2^{-\mathfrak{q}t_i}\xi_{i-1},\ 2^{-t_i}\xi_i,\ 2^{-\mathfrak{q}t_i}\xi_{i+1}, \ldots, 2^{-\mathfrak{q}t_i}\xi_n\right)\widehat{f}(\xi).\end{aligned} \tag{3. 7}$$

For each n-tuple $\mathbf{t}$: (**H**) *There exists at least one* $i \in \{1, 2, \ldots, n\}$ *such that*

$$t_i \geq \frac{1}{\mathfrak{q}-1}\left(2 + \log_2 n\right). \tag{3. 8}$$

Lemma 3.1 *Let* $\mathbf{t}$ *satisfying* ($\mathbf{H}$) *in (3. 8) and* $0 < \rho < 1$. *Consider* $\mathbf{I} \cup \mathbf{J} = \{1, 2, \ldots, n\}$ *such that*

$$\Big(t_j + 2 + \log_2 n\Big)/q \;\leq\; t_i \;\leq\; q t_j - \Big(2 + \log_2 n\Big) \qquad \textit{for all } \; i, j \in \mathbf{I} \tag{3. 9}$$

and

$$q t_j - \Big(2 + \log_2 n\Big) \;<\; t_\imath \;=\; \max\Big(t_i : i \in \mathbf{I}\Big) \qquad \textit{for all } \; j \in \mathbf{J}. \tag{3. 10}$$

For $\xi \in \mathbf{supp}\delta_{\mathbf{t}}(\xi)$, *we have*

$$|\xi_i| \;\sim\; 2^{t_i} \quad \textit{for every } \; i \in \mathbf{I} \tag{3. 11}$$

and

$$|\xi_j| \;\lesssim\; 2^{t_j}, \qquad |\xi_\imath| \;\sim\; 2^{t_\imath} \;\sim\; 2^{q t_j} \quad \textit{for every } \; j \in \mathbf{J}. \tag{3. 12}$$

Proof : We write

$$\xi \;=\; \Big(\xi_i, \xi_i^\dagger\Big) \;\in\; \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}^{\mathbf{N} - \mathbf{N}_i}, \qquad i = 1, 2, \ldots, n. \tag{3. 13}$$

Let $\xi \in \mathbf{supp}\delta_{\mathbf{t}}(\xi)$. By definition of $\delta_{\mathbf{t}}(\xi)$ in (3. 5)-(3. 6), we first have

$$\begin{aligned} |\xi_i| \;&<\; 2^{t_i+1}, \qquad i = 1, 2, \ldots, n \;\text{ and} \\ |\xi_i| \;&<\; 2^{q t_j+1}, \qquad \text{for } \; j \neq i. \end{aligned} \tag{3. 14}$$

On the other hand, we either have

$$|\xi_i| \;>\; \frac{2^{t_i-1}}{\sqrt{2}} \qquad \text{or} \qquad \left|\xi_i^\dagger\right| \;>\; \frac{2^{q t_i-1}}{\sqrt{2}} \tag{3. 15}$$

for every $i = 1, 2, \ldots, n$.

Let $i \in \mathbf{I}$ in (3. 9). If ξ_i does not satisfy the first inequality in (3. 15), there exists at least one ξ_j for some $j \neq i$ such that

$$|\xi_j| \;>\; \frac{2^{q t_i-1}}{\sqrt{2}} \frac{1}{\sqrt{n-1}}. \tag{3. 16}$$

Together with the first inequality in (3. 14), we must have

$$q t_i - 2 - \frac{1}{2}\log_2 2(n-1) \;<\; t_j. \tag{3. 17}$$

Suppose $j \in \mathbf{I}$ in (3. 9). We have

$$t_j \;\leq\; q t_i - \Big(2 + \log_2 n\Big). \tag{3. 18}$$

The inequality in (3. 17) cannot hold because

$$(1/2)\log_2 2(n-1) \;<\; \log_2 n \qquad \text{for } \; n \geq 2. \tag{3. 19}$$

Suppose $j \in \mathbf{J}$ in (3. 10). Since $i \in \mathbf{I}$ in (3. 9), we have

$$\begin{aligned} q t_i - \Big(2 + \log_2 n\Big) - t_j \;&>\; t_\imath - \frac{1}{q}\Big(t_\imath + \Big(2 + \log_2 n\Big)\Big) \\ &=\; t_\imath\Big(1 - \frac{1}{q}\Big) - \frac{1}{q}\Big(2 + \log_2 n\Big) \\ &\geq\; \left[\Big(\frac{q-1}{q}\Big)\Big(\frac{1}{q-1}\Big) - \frac{1}{q}\right]\Big(2 + \log_2 n\Big) \\ &=\; 0 \end{aligned} \tag{3. 20}$$

where the second inequality follows from (**H**) in (3. 8). Notice that (3. 20) implies (3. 18) again. Therefore, ξ_i necessarily satisfies the first inequality in (3. 15). Together with the first inequality in (3. 14), we have $|\xi_i| \sim 2^{t_i}$ for every $i \in \mathbf{I}$ in (3. 9). On the other hand, from (3. 10) and the second inequality in (3. 14), we have $|\xi_\imath| \sim 2^{t_\imath} \sim 2^{q t_j}$ for every $j \in \mathbf{J}$. □

Remark 3.1 *Let* $\mathbf{I} \cup \mathbf{J} = \{1, 2, \ldots, n\}$ *defined with respect to (3. 9)-(3. 10) and* $t_\imath = \max(t_i : i \in \mathbf{I})$*. For* $\xi \in \mathbf{supp}\,\delta_{\mathbf{t}}(\xi)$*, Lemma 3.1 implies*

$$|\xi| \sim |\xi_\imath| \sim 2^{t_\imath} \sim 2^{q t_j} \lesssim 2^{q t_i}, \qquad i \in \mathbf{I} \text{ and } j \in \mathbf{J}. \tag{3. 21}$$

Lemma 3.2 *For every given* $\mathbf{t}$*, we can partition the set* $\{1, 2, \ldots, n\}$ *into* $\mathbf{I} \cup \mathbf{J}$ *such that (3. 9)-(3. 10) hold respectively.*

Proof: Consider π to be a permutation acting on the set $\{1, 2, \ldots, n\}$. We can assume that $t_{\pi(1)} \le t_{\pi(2)} \le \cdots \le t_{\pi(n)}$. Let $k \in \{1, 2, \ldots, n\}$ such that

$$t_{\pi(k)} = \min\Big(\pi(i) : t_{\pi(n)} < q t_{\pi(i)} - \big(2 + \log_2 n\big)\Big). \tag{3. 22}$$

We thus define

$$\mathbf{I} = \Big\{\pi(k), \pi(k+1), \ldots, \pi(n)\Big\} \text{ and } \mathbf{J} = \Big\{\pi(1), \pi(2), \ldots, \pi(k-1)\Big\}. \tag{3. 23}$$

□

Lemma 3.3 *Let* $\sigma \in \mathbf{S}^0_\rho$*. Suppose* $\mathbf{t}$ *satisfying (***H***) in (3. 8). We have*

$$\left|\prod_{i=1}^n \left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i} \delta_{\mathbf{t}}(\xi)\sigma(x,\xi)\right| \le A_\alpha \prod_{i=1}^n 2^{-t_i|\alpha_i|} \tag{3. 24}$$

for every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$*.*

Proof: Certainly, for every $\mathbf{t}$ satisfying (**H**) in (3. 8), $\delta_{\mathbf{t}}(\xi)$ defined in (3. 5)-(3. 6) satisfies the differential inequality in (3. 24). On the other hand, $\sigma \in \mathbf{S}^0_\rho$ satisfies the differential inequality in (1. 3) for $m = 0$. At each i-th component, we have

$$\left|\left(\frac{\partial}{\partial \xi_i}\right)^{\alpha_i} \sigma(x,\xi)\right| \lesssim \left(\frac{1}{1 + |\xi_i| + |\xi|^\rho}\right)^{|\alpha_i|} \tag{3. 25}$$

for $i = 1, 2, \ldots, n$.

Let $0 < \rho < 1$. From Lemma 3.1 and Remark 3.1, we either have

$$|\xi_i| \sim 2^{t_i}, \ |\xi| \lesssim 2^{t_i/\rho} \text{ for } i \in \mathbf{I} \qquad \text{or} \qquad |\xi_j| \lesssim 2^{t_j}, \ |\xi| \sim 2^{t_j/\rho} \text{ for } j \in \mathbf{J}. \tag{3. 26}$$

Therefore, we have

$$1 + |\xi_i| + |\xi|^\rho \sim 2^{t_i} \tag{3. 27}$$

for every $i = 1, 2, \ldots, n$.

When $\rho = 0$, by definition of $\delta_{\mathbf{t}}$ in (3. 5)-(3. 6), we have

$$|\xi^i| \sim 2^{t_i} \tag{3. 28}$$

for every $i = 1, 2, \ldots, n$. □

4 Classification of Symbols and Kernels

Suppose that $\sigma(x,\xi) \in \mathbf{S}_\rho$ satisfies the differential inequality in (1. 3). Let $z = x - y$ and $\Omega(x,y) = \Omega(x,z)$ in (1. 5). We have

$$\begin{aligned} \left(\frac{\partial}{\partial z}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}\Omega(x,z) &= \left(\frac{1}{2\pi\mathbf{i}}\right)^{-|\alpha|}\int e^{2\pi\mathbf{i}z\cdot\xi}\prod_{i=1}^{n}(\xi_i)^{\alpha_i}\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\sigma(x,\xi)\right)d\xi \\ &= \left(\frac{1}{2\pi\mathbf{i}}\right)^{-|\alpha|}\sum_{\mathbf{t}}\int e^{2\pi\mathbf{i}z\cdot\xi}\prod_{i=1}^{n}(\xi_i)^{\alpha_i}\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\delta_{\mathbf{t}}(\xi)\sigma(x,\xi)\right)d\xi. \end{aligned} \tag{4. 1}$$

It is suffice to take the summation in (4. 1) over all $\mathbf{t}$ satisfying $(\mathbf{H})$ in (3. 8). By definition of $\delta_{\mathbf{t}}$ in (3. 5)-(3. 6), the remaining term has an integrant compactly supported for $0 \leq \rho < 1$, and belongs to $C^\infty\left(\mathbb{R}^{\mathbf{N}}\right)$ with rapidly decays in all derivatives as $|z| \longrightarrow \infty$.

Let $\xi \in \mathbf{supp}\delta_{\mathbf{t}}(\xi)$. By Lemma 3.3, every ∂_{ξ_i} acting on $\delta_{\mathbf{t}}(\xi)\sigma(x,\xi)$ gains a constant multiple of 2^{-t_i} for every $i = 1, 2, \ldots, n$. First, let $0 < \rho < 1$. By Lemma 3.1, every ∂_x acting on $\delta_{\mathbf{t}}(\xi)\sigma(x,\xi)$ gains a constant multiple of $2^{\rho t_\imath}$, whereas $|\xi| \sim |\xi_\imath| \sim 2^{t_\imath}$ as discussed in Remark 3.1.

Consider the norm of

$$\begin{aligned} &(z_i)^{\gamma_i}(z_j)^{\gamma_j}\int_{\mathbb{R}^{\mathbf{N}_i}} e^{2\pi\mathbf{i}z\cdot\xi}\delta_{\mathbf{t}}(\xi)\,(\xi_i)^{\alpha_i}\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\sigma(x,\xi)\right)d\xi_i \\ &= \left(\frac{-1}{2\pi\mathbf{i}}\right)^{|\gamma_i|+|\gamma_j|}\int_{\mathbb{R}^{\mathbf{N}_i}} e^{2\pi\mathbf{i}z\cdot\xi}\left(\frac{\partial}{\partial\xi_i}\right)^{\gamma_i}\left(\frac{\partial}{\partial\xi_j}\right)^{\gamma_j}\left\{\delta_{\mathbf{t}}(\xi)\,(\xi_i)^{\alpha_i}\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\sigma(x,\xi)\right)\right\}d\xi_i \end{aligned} \tag{4. 2}$$

for every multi-index $\gamma_i, \gamma_j, \alpha_i$ and β.

Suppose that $i = \imath$. The norm of integral in (4. 2) is bounded by a constant multiple of

$$2^{-|\gamma_i|t_i-|\gamma_j|t_j+\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|\right)t_i}. \tag{4. 3}$$

By Lemma 3.1, we have $t_i \lesssim t_j/\rho$ for every pair of $i,\ j \in \{1, 2, \ldots, n\}$. Therefore, the norm of integral in (4. 2) can be further bounded by a constant multiple of

$$2^{-\left(|\gamma_i|+\rho|\gamma_j|\right)t_i} \times 2^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|\right)t_i}. \tag{4. 4}$$

Let $|\gamma_j|$ to be fixed, we consider separately for $\displaystyle\sum_{2^{t_i}\leq|z_i|^{-1}}$ and $\displaystyle\sum_{2^{t_i}>|z_i|^{-1}}$.

For $2^{t_i} \leq |z_i|^{-1}$, we choose $|\gamma_i| = 0$ so that

$$\sum_{2^{t_i}\leq|z_i|^{-1}} 2^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-\rho|\gamma_j|\right)t_i} \lesssim \left(\frac{1}{|z_i|}\right)^{\mathbf{N}_i+|\alpha_i|+\rho|\beta|-\rho|\gamma_j|}. \tag{4. 5}$$

For $2^{t_i} > |z_i|^{-1}$, we choose $|\gamma_i| > \mathbf{N}_i + |\alpha_i| + \rho|\beta| - \rho|\gamma_j|$ so that

$$\left(\frac{1}{|z_i|}\right)^{|\gamma_i|}\sum_{2^{t_i}>|z_i|^{-1}} 2^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-\rho|\gamma_j|-|\gamma_i|\right)t_i} \lesssim \left(\frac{1}{|z^i|}\right)^{\mathbf{N}_i+|\alpha_i|+\rho|\beta|-\rho|\gamma_j|}. \tag{4. 6}$$

On the other hand, let $|\gamma_i|$ to be fixed. We consider $\displaystyle\sum_{2^{t_i}\leq|z_j|^{-1/\rho}}$ and $\displaystyle\sum_{2^{t_i}>|z_j|^{-1/\rho}}$ separately.

For $2^{t_i}\leq|z_j|^{-1/\rho}$, we choose $|\gamma_j|=0$ so that

$$\sum_{2^{t_i}\leq|z_j|^{-1/\rho}} 2^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-|\gamma_i|\right)t_i} \lesssim \left(\frac{1}{|z_j|}\right)^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-|\gamma_i|\right)/\rho}. \tag{4. 7}$$

For $2^{t_i}>|z_j|^{-1/\rho}$, we choose $|\gamma_j|>\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-|\gamma_i|\right)/\rho$ so that

$$\left(\frac{1}{|z_j|}\right)^{|\gamma_j|}\sum_{2^{t_i}>|z_j|^{-1/\rho}} 2^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-|\gamma_i|-\rho|\gamma_j|\right)t_i} \lesssim \left(\frac{1}{|z_j|}\right)^{\left(\mathbf{N}_i+|\alpha_i|+\rho|\beta|-|\gamma_i|\right)/\rho}. \tag{4. 8}$$

By carrying out the estimation in (4. 2) for all other coordinate subspaces, and taking into account that the component $\imath$ varies from 1 to n for different $\mathbf{t}$, we obtain

$$\prod_{i=1}^{n}|z_i|^{|\gamma_i|}\left|z_j\right|^{|\gamma_j|}\left|\left(\frac{\partial}{\partial z}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}\Omega(x,z)\right| \leq A_{\alpha\beta\gamma} \tag{4. 9}$$

for every multi-index α and β, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$, provide that

$$|\gamma_i|+\rho|\gamma_j| \geq \mathbf{N}_i+|\alpha_i|+\rho|\beta| \tag{4. 10}$$

for every $i,j=1,2,\ldots,n$.

For $\rho=0$, we carry out the integration by parts in (4. 2) with $\gamma^j=0$. The estimate in (4. 3) holds for every $i=1,2,\ldots,n$. From (4. 5)-(4. 6), the estimates in (4. 9)-(4. 10) remain valid. We can then verify the differential inequality in (1. 6).

To show the cancellation properties in (1. 7), we write

$$\begin{aligned}&\int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\Omega(x,z)\right)\prod_{i\in\mathbf{I}}\varphi_i\left(R_iz_i\right)dz_i\\ &= \int e^{2\pi\mathbf{i}z_{\mathbf{J}}\cdot\xi_{\mathbf{J}}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\sigma(x,\xi)\right)\prod_{i\in\mathbf{I}}R_i^{-\mathbf{N}_i}\widehat{\varphi}_i\left(-R_i^{-1}\xi_i\right)d\xi.\end{aligned} \tag{4. 11}$$

Observe that (4. 11) equals

$$\int_{\bigoplus_{j\in\mathbf{J}}\mathbb{R}^{\mathbf{N}_j}} e^{2\pi\mathbf{i}z_{\mathbf{J}}\cdot\xi_{\mathbf{J}}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\zeta\left(x,\xi_{\mathbf{J}}\right)\right)d\xi_{\mathbf{J}} \tag{4. 12}$$

where

$$\begin{aligned}\left(\frac{\partial}{\partial x}\right)^{\alpha}\zeta\left(x,\xi_{\mathbf{J}}\right) &= \left(\frac{\partial}{\partial x}\right)^{\alpha}\int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}}\sigma(x,\xi)\prod_{i\in\mathbf{I}}R_i^{-\mathbf{N}_i}\widehat{\varphi}_i\left(-R_i^{-1}\xi_i\right)d\xi_i\\ &= \int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\sigma(x,\xi)\right)\prod_{i\in\mathbf{I}}R_i^{-\mathbf{N}_i}\widehat{\varphi}_i\left(-R_i^{-1}\xi_i\right)d\xi_i.\end{aligned} \tag{4. 13}$$

For the given multi-index α, define the norm function

$$\vartheta_\alpha(\xi) \ = \ \sum_{i\in\mathbf{I}} |\xi_i|^{\rho|\alpha|} \ + \ \Big(1+\big|\xi_{\mathbf{J}}\big|\Big)^{\rho|\alpha|} \ \sim \ (1+|\xi|)^{\rho|\alpha|}. \tag{4. 14}$$

Rewrite the integral in (4. 13) by

$$\begin{gathered}\int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}} |\xi_i|^{\rho|\alpha|}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\sigma(x,\xi)\vartheta_\alpha^{-1}(\xi)\right)\prod_{i\in\mathbf{I}} R_i^{-\mathbf{N}_i}\widehat{\varphi}_i\left(-R_i^{-1}\xi_i\right)d\xi_i \\ + \ \Big(1+\big|\xi_{\mathbf{J}}\big|\Big)^{\rho|\alpha|}\int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}}\left(\left(\frac{\partial}{\partial x}\right)^{\alpha}\sigma(x,\xi)\vartheta_\alpha^{-1}(\xi)\right)\prod_{i\in\mathbf{I}} R_i^{-\mathbf{N}_i}\widehat{\varphi}_i\left(-R_i^{-1}\xi_i\right)d\xi_i.\end{gathered} \tag{4. 15}$$

Observe that $\big(\partial_x^\alpha\sigma\big)(x,\xi)\vartheta_\alpha^{-1}(\xi)$ is bounded and satisfies the differential inequality in (1. 3) provided that $\sigma\in\mathbf{S}^0_\rho$. By letting $\xi_i\longrightarrow R_i\xi_i$, the sufficient smoothness of φ_i for $i\in\mathbf{I}$ implies that the norm of every integral in the sum of (4. 15) is bounded by $A_\alpha(R_i)^{\rho|\alpha|}$. By carrying out the same estimation developed in step **1**, on the subspace $\bigoplus_{j\in\mathbf{J}}\mathbb{R}^{\mathbf{N}_j}$ with $\sigma(x,\xi)$ replaced by $\zeta(x,\xi_{\mathbf{J}})$ given implicitly in (4. 13), we obtain the differential inequality in (1. 7).

Suppose that Ω in (1. 5) satisfies (1. 6)-(1. 7). Define

$$\rho_i(\xi) \ = \ 1+|\xi_i|+|\xi|^{\rho} \tag{4. 16}$$

for $0\le\rho<1$ and every $i=1,2,\ldots,n$. We show $\sigma\in\mathbf{S}^0_\rho$ by proving that the norm of

$$\begin{aligned}&\prod_{i=1}^n \rho_i(\xi)^{|\alpha_i|}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\int\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\Omega(x,z)\right)e^{-2\pi\mathbf{i}z\cdot\xi}dz \\ &= \ (-1)^{|\alpha|}(2\pi\mathbf{i})^{|\alpha|}\int\prod_{i=1}^n\Big(\rho_i(\xi)z_i\Big)^{\alpha_i}\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\Omega(x,z)\right)e^{-2\pi\mathbf{i}z\cdot\xi}dz\end{aligned} \tag{4. 17}$$

is bounded by $A_{\alpha\beta}\,(1+|\xi|)^{\rho|\beta|}$ for every multi-index $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and β.

Let φ_i to be the normalized bump function for every $i\in\{1,2,\ldots,n\}$. (4. 17) consists of

$$\begin{aligned}\int\cdots\int_{\bigoplus_{j\in\mathbf{J}}\mathbb{R}^{\mathbf{N}_j}}&\left\{\int\cdots\int_{\bigoplus_{i\in\mathbf{I}}\mathbb{R}^{\mathbf{N}_i}}\left(\left(\frac{\partial}{\partial x}\right)^{\beta}\Omega(x,z)\right)\prod_{i\in\mathbf{I}}\Big(\rho_i(\xi)z_i\Big)^{\alpha_i}\varphi_i\Big(\rho_i(\xi)|z_i|\Big)e^{2\pi\mathbf{i}z_i\cdot\xi_i}dz_i\right\} \\ &\times\prod_{j\in\mathbf{J}}e^{2\pi\mathbf{i}z_j\cdot\xi_j}\Big(\rho_j(\xi)z_j\Big)^{\alpha_j}\Big(1-\varphi_j\Big(\rho_j(\xi)|z_j|\Big)\Big)dz_j\end{aligned} \tag{4. 18}$$

where $\mathbf{I}\cup\mathbf{J}=\{1,2,\ldots,n\}$. Observe that (4. 18) vanishes when $|z_{\mathbf{J}}|$ is small. On the other hand,

$$(z_i)^{\alpha_i}\varphi_i(|z_i|)\mathbf{exp}\left(2\pi\mathbf{i}\left(\frac{z_i\cdot\xi_i}{\rho_i(\xi)}\right)\right), \qquad i\in\mathbf{I} \tag{4. 19}$$

is an normalized bump function. Suppose that $|\xi|$ is small. By (1. 7), where $\Omega(x,z)$ decays rapidly as $|z_{\mathbf{J}}|\longrightarrow\infty$, the norm of integral in (4. 18) is bounded by

$$A_{\alpha\beta}\left(1+\sum_{i\in\mathbf{I}}\rho_i(\xi)\right)^{\rho|\beta|} \ \lesssim \ A_{\alpha\beta}\left(1+\sum_{i=1}^n\rho_i(\xi)\right)^{\rho|\beta|} \ \lesssim \ A_{\alpha\beta}\,(1+|\xi|)^{\rho|\beta|}. \tag{4. 20}$$

Suppose that $|\xi|$ is large. Let $\xi_\imath$ to be the largest component of ξ. We carry out an N-fold integration by parts with respect to $z_\imath$ inside the integral of (4. 18). Notice that the boundary terms vanish provided that it is an normalized bump function in (4. 19) if $\imath \in \mathbf{I}$ and $\Omega(x, z)$ satisfies the cancellation properties in (1. 7) if $\imath \in \mathbf{J}$. If the differentiation falls on any of $1 - \varphi_j$ for $j \in \mathbf{J}$, its derivative is another normalized bump function. By letting N to be sufficiently large, the resulting term has its norm bounded as (4. 20).

5 Boundedness of Partial Sum Operators

Let $\mathbf{M}$ to be the strong maximal function operator. Recall the partial sum operator $\Delta_{\mathbf{t}}$ defined in (3. 7).

Lemma 5.1 *Let $\sigma \in \mathbf{S}_\rho$. We have*

$$\Big|\Big(\Delta_{\mathbf{t}} T_\sigma f\Big)(x)\Big| \lesssim \Big(\mathbf{M} f\Big)(x) \tag{5. 1}$$

for every $\mathbf{t}$ satisfying ($\mathbf{H}$) in (3. 8).

Proof : A direct computation shows

$$\begin{aligned} \Big(\Delta_{\mathbf{t}} T_\sigma f\Big)(x) &= \int f(y)\Omega_{\mathbf{t}}(x, y)dy \\ &= \int f(y)\left\{\int e^{2\pi\mathbf{i}(x-y)\cdot\xi}\delta_{\mathbf{t}}(\xi)\Lambda(y, \xi)d\xi\right\}dy \end{aligned} \tag{5. 2}$$

where

$$\Lambda(x, \xi) = \iint e^{2\pi\mathbf{i}(x-y)\cdot(\xi-\eta)}\sigma(y, \eta)dyd\eta. \tag{5. 3}$$

We have $\sigma \in \mathbf{S}^0_\rho$ bounded and satisfies the differential inequality in (2. 1). By Lemma 2.1, $\Lambda(x, \xi)$ is bounded and satisfies the differential inequality in (2. 3).

Let $z = x - y$. By changing dilations $\xi \longrightarrow \mathbf{t}^{-1}\xi$ and $z \longrightarrow \mathbf{t}z$, the integral in (5. 2) can be written as

$$\int f(x - \mathbf{t}z)\left\{\int e^{2\pi\mathbf{i}z\cdot\xi}\delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right)\Lambda(x - \mathbf{t}z, \mathbf{t}^{-1}\xi)d\xi\right\}dz. \tag{5. 4}$$

Recall that $\mathfrak{q} = 1/\rho$ for $0 \leq \rho < 1$. By definition of $\delta_{\mathbf{t}}$ in (3. 5)-(3. 6), the support of

$$\delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right) = \prod_{i=1}^{n}\phi\left(2^{t_1-\mathfrak{q}t_i}\xi_1, \ldots, 2^{t_{i-1}-\mathfrak{q}t_i}\xi_{i-1},\ \xi_i,\ 2^{t_{i+1}-\mathfrak{q}t_i}\xi_{i+1}, \ldots, 2^{t_n-\mathfrak{q}t_i}\xi_n\right) \tag{5. 5}$$

lies inside a ball with radius 2. Hence that the kernel in (5. 4) is bounded. Moreover, from Lemma 3.1 we have $t_i \lesssim \mathfrak{q}t_j$ for every $i, j \in \{1, 2, \ldots, n\}$ whenever the support is nonempty.

For every given $\mathbf{t}$, from Lemma 3.2, we can partition the set $\{1, 2, \ldots, n\}$ into $\mathbf{I} \cup \mathbf{J}$ with respect to (3. 9)-(3. 10). Integration by parts with respect to ξ in (5. 4) gives

$$\begin{aligned} &\left(\frac{1}{2\pi \mathbf{i} z}\right)^{\alpha} \int e^{2\pi \mathbf{i} z\cdot\xi} \left\{\left(\frac{\partial}{\partial\xi}\right)^{\alpha} \delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right) \Lambda(x - \mathbf{t}z, \mathbf{t}^{-1}\xi)\right\} d\xi \\ = &\left(\frac{1}{2\pi \mathbf{i} z}\right)^{\alpha} \int e^{2\pi \mathbf{i} z\cdot\xi} \left\{\prod_{i\in\mathbf{I}} \left(\frac{\partial}{\partial\xi_i}\right)^{\alpha_i} \prod_{j\in\mathbf{J}} \left(\frac{\partial}{\partial\xi_j}\right)^{\alpha_j} \delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right) \Lambda(x - \mathbf{t}z, \mathbf{t}^{-1}\xi)\right\} d\xi \end{aligned} \tag{5. 6}$$

at $z_i \neq 0$ for every $i = 1, 2, \ldots, n$ and every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

Recall that $\Lambda(x, \xi)$ defined in (5. 3) satisfies the differential inequality in (2. 3). By Lemma 3.1, we have $|\xi_i| \sim 1$ for $i \in \mathbf{I}$ whenever $\xi \in \mathbf{supp}\delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right)$. Therefore

$$\begin{aligned} &\left|\prod_{i\in\mathbf{I}} \left(\frac{\partial}{\partial\xi_i}\right)^{\alpha_i} \prod_{j\in\mathbf{J}} \left(\frac{\partial}{\partial\xi_j}\right)^{\alpha_j} \delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right) \Lambda(x - \mathbf{t}z, \mathbf{t}^{-1}\xi)\right| \\ &\qquad \lesssim \prod_{i\in\mathbf{I}} \left(2^{t_i}\right)^{|\alpha_i|} \left(\frac{1}{1 + 2^{t_i}|\xi_i| + |\mathbf{t}^{-1}\xi|^{\rho}}\right)^{|\alpha_i|} \prod_{j\in\mathbf{J}} \left(2^{t_j}\right)^{|\alpha_j|} \left(\frac{1}{1 + 2^{t_j}|\xi_j| + |\mathbf{t}^{-1}\xi|^{\rho}}\right)^{|\alpha_j|} \\ &\qquad \lesssim \prod_{i\in\mathbf{I}} \left(2^{t_i}\right)^{|\alpha_i|} \left(\frac{1}{1 + 2^{t_i}|\xi_i|}\right)^{|\alpha_i|} \prod_{j\in\mathbf{J}} \left(2^{t_j}\right)^{|\alpha_j|} \left(\frac{1}{1 + (2^{t_\imath})^{\rho}|\xi_\imath|^{\rho}}\right)^{|\alpha_j|} \\ &\qquad \lesssim \prod_{i\in\mathbf{I}} \left(\frac{1}{|\xi_i|}\right)^{|\alpha_i|} \prod_{j\in\mathbf{J}} \left(\frac{1}{|\xi_\imath|}\right)^{\rho|\alpha_j|} \\ &\qquad \leq A_{\alpha\beta} \end{aligned} \tag{5. 7}$$

for $0 < \rho < 1$ and every multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

At $\rho = 0$, the estimate in (5. 7) remains valid, provided that $|\xi_i| \sim 1$ for every $i = 1, 2, \ldots, n$ whenever $\mathbf{t}^{-1}\xi \in \mathbf{supp}\delta_{\mathbf{t}}\left(\mathbf{t}^{-1}\xi\right)$.

From (5. 6)-(5. 7), we have

$$\left|\left(\Delta_{\mathbf{t}} T_{\sigma} f\right)(x)\right| \leq A_N \int |f(x - \mathbf{t}z)| \left(\frac{1}{1 + |z|}\right)^N dz \tag{5. 8}$$

for every $N \geq 1$. The function $\left(1 + |z|\right)^{-N}$ can be approximated by $\sum_k a_k \chi_{\mathbf{B}_k}$ where each a_k is a positive constant and $\mathbf{B}_k$ is a standard ball in $\mathbb{R}^{\mathbf{N}}$ centred on the origin, for which $\sum_k a_k |\mathbf{B}_k| < \infty$ provided that N is sufficiently large.

From (5. 8), we have

$$\int |f(x - \mathbf{t}z)| \left(\sum_k a_k \chi_{\mathbf{B}_k}\right) dz \lesssim \left(\sum_k a_k |\mathbf{B}_k|\right) \sup_k \frac{1}{|\mathbf{B}_k(\mathbf{t}^{-1}x)|} \int_{\mathbf{B}_k(\mathbf{t}^{-1}x)} |f(y)| dy \tag{5. 9}$$

where the right hand side of (5. 9) is bounded by a constant multiple of $(\mathbf{M}f)(x)$. □

6 Decaying Estimate of Partial Sum Operators

Let $\mathbf{s}$ to be the n-tuple defined as same as $\mathbf{t}$ in (3. 1)-(3. 4). Suppose that T_σ is translation invariant, i.e: $\widehat{T_\sigma f}(\xi) = \sigma(\xi)\widehat{f}(\xi)$. Then, $\Delta_\mathbf{t}$ is commute with T_σ. From (3. 5)-(3. 6), we have $\Delta_\mathbf{t} T_\sigma \Delta_\mathbf{s} = 0$ for $|\mathbf{t}-\mathbf{s}| = \sum_i |t_i - s_i| > \mathbf{const}$.

Lemma 6.1 *Suppose that* $\sigma \in \mathbf{S}_\rho$, *we have*

$$\left|\Big(\Delta_\mathbf{t} T_\sigma \Delta_\mathbf{s} f\Big)(x)\right| \lesssim \prod_{i=1}^n 2^{-(1-\rho)|t_i-s_i|}\Big(\mathbf{M}f\Big)(x) \tag{6. 1}$$

for every $\mathbf{t}$ *and* $\mathbf{s}$ *satisfying* ($\mathbf{H}$) *in (3. 8).*

Proof: 1. Recall the formulae in (5. 2)-(5. 3). By direct computations, we have

$$\begin{aligned}\Big(\Delta_\mathbf{t} T_\sigma \Delta_\mathbf{s} f\Big)(x) &= \int f(y)\Omega_{\mathbf{ts}}(x,y)dy \\ &= \int f(y)\left\{\int e^{2\pi \mathbf{i}(x-y)\cdot\xi}\delta_\mathbf{t}(\xi)\Lambda_\mathbf{s}(y,\xi)d\xi\right\}dy\end{aligned} \tag{6. 2}$$

where

$$\Lambda_\mathbf{s}(x,\xi) = \iint e^{2\pi\mathbf{i}(x-y)\cdot(\xi-\eta)}\delta_\mathbf{s}(\eta)\sigma(y,\eta)dyd\eta. \tag{6. 3}$$

Suppose that $\xi_i \neq \eta_i$ for some $i = 1,2,\ldots,n$. Integration by parts with respect to y_i gives

$$\Lambda_\mathbf{s}(x,\xi) = \left(\frac{-1}{4\pi^2}\right)\iint e^{2\pi\mathbf{i}(x-y)\cdot(\xi-\eta)}\delta_\mathbf{s}(\eta)\left(\frac{1}{|\xi_i-\eta_i|}\right)^2\Big(\Delta_{y_i}\sigma(y,\eta)\Big)dyd\eta. \tag{6. 4}$$

Since $\sigma \in \mathbf{S}_\rho$, we have

$$\left|\left(\frac{1}{|\xi_i-\eta_i|}\right)^2\Big(\Delta_{y_i}\sigma(y,\eta)\Big)\right| \lesssim \left(\frac{|\eta|^\rho}{|\xi_i-\eta_i|}\right)^2 \tag{6. 5}$$

where

$$\xi \in \mathbf{supp}\delta_\mathbf{t}(\xi) \qquad \text{and} \qquad \eta \in \mathbf{supp}\delta_\mathbf{s}(\eta). \tag{6. 6}$$

We shall assume that there exists at least one $i = \ell \in \{1,2,\ldots,n\}$ such that $|t_\ell - s_\ell|$ is large. Let $\rho = 0$. From (3. 5)-(3. 6), we have $|\xi_i| \sim 2^{t_i}$ and $|\eta_i| \sim 2^{s_i}$ for every $i = 1,2,\ldots,n$. Therefore, $|\xi_\ell - \eta_\ell| \sim 2^{\max(t_\ell,s_\ell)}$. It follows that $\delta_\mathbf{s}(\eta)\Big(\Delta_{y_\ell}\sigma\Big)(y,\eta)/|\xi_\ell-\eta_\ell|^2$ satisfies the differential inequality in (2. 1) with its norm bounded by a constant multiple of $2^{-|t_\ell-s_\ell|}$.

2: Let $0<\rho<1$ and $\mathfrak{q} = 1/\rho$. By Lemma 3.2, for given $\mathbf{t}$ and $\mathbf{s}$, set $\mathbf{I}_1 \cup \mathbf{J}_1 = \mathbf{I}_2 \cup \mathbf{J}_2 = \{1,2,\ldots,n\}$ with respect to (3. 9)-(3. 10). We write t_i for every $i \in \mathbf{I}_1 \cup \mathbf{J}_1$ and s_i for every $i \in \mathbf{I}_2 \cup \mathbf{J}_2$ respectively. Let $t_\imath = \max\{t_i : i \in \mathbf{I}_1\}$ and $s_\jmath = \max\{s_i : i \in \mathbf{I}_2\}$.

Suppose $\ell \in \mathbf{I}_1 \cap \mathbf{I}_2$. By Lemma 3.1, we have $|\xi_\ell| \sim 2^{t_\ell}$ and $|\eta_\ell| \sim 2^{s_\ell}$. Therefore

$$|\xi_\ell - \eta_\ell| \sim \begin{cases} 2^{t_\ell} = 2^{s_\ell}2^{(t_\ell-s_\ell)}, & t_\ell > s_\ell, \\ 2^{s_\ell} = 2^{t_\ell}2^{(s_\ell-t_\ell)}, & t_\ell < s_\ell. \end{cases} \tag{6. 7}$$

From Remark 3.1, the first equality in (6. 7) implies

$$\frac{|\eta|}{|\xi_\ell-\eta_\ell|^{\mathfrak{q}}} ~\lesssim~ 2^{-\mathfrak{q}|t_\ell-s_\ell|}. \tag{6. 8}$$

Suppose $t_\ell < s_\ell$ and $\mathfrak{q}t_\ell \geq s_\jmath$, the second equality in (6. 7) implies (6. 8) as well. If $t_\ell < s_\ell$ and $\mathfrak{q}t_\ell < s_\jmath$, Remark 3.1 implies $|\xi| \lesssim |\eta|$. By replacing ℓ with $\jmath$ in (6. 4), we have

$$\frac{|\eta|}{|\xi_\jmath-\eta_\jmath|^{\mathfrak{q}}} ~\lesssim~ 2^{-(\mathfrak{q}-1)s_\jmath} ~\lesssim~ 2^{-(\mathfrak{q}-1)s_\ell}, \qquad t_\ell < s_\ell. \tag{6. 9}$$

Suppose $\ell \in \mathbf{I}_1 \cap \mathbf{J}_2$. By Lemma 3.1, we have $|\xi_\ell| \sim 2^{t_\ell}$ and $|\eta_\ell| \lesssim 2^{s_\ell}$. If $t_\ell > s_\ell$, the first equality in (6. 7) is valid which implies (6. 8). If $t_\ell < s_\ell$, Remark 3.1 implies $|\xi| \lesssim |\eta|$. By replacing ℓ with $\jmath$ in (6. 4), the estimate in (6. 9) follows.

Suppose $\ell \in \mathbf{J}_1 \cap \mathbf{I}_2$. By Lemma 3.1, we have $|\xi_\ell| \lesssim 2^{t_\ell}$ and $|\eta_\ell| \sim 2^{s_\ell}$. If $t_\ell > s_\ell$, Remark 3.1 implies $|\xi| \gtrsim |\eta|$. By replacing ℓ with $\imath$ in (6. 4), we have

$$\frac{|\eta|}{|\xi_\imath-\eta_\imath|^{\mathfrak{q}}} ~\lesssim~ 2^{-(\mathfrak{q}-1)t_\imath} ~\lesssim~ 2^{-(\mathfrak{q}-1)t_\ell}, \qquad t_\ell > s_\ell. \tag{6. 10}$$

If $t_\ell < s_\ell$ and $\mathfrak{q}t_\ell \geq s_\jmath$, the second equality in (6. 7) is valid which implies (6. 8). When $t_\ell < s_\ell$ and $\mathfrak{q}t_\ell < s_\jmath$, Remark 3.1 implies $|\xi| \lesssim |\eta|$. By replacing ℓ with $\jmath$ in (6. 4), the estimate in (6. 9) follows.

Suppose $\ell \in \mathbf{J}_1 \cap \mathbf{J}_2$. Remark 3.1 implies $|\xi| \sim 2^{\mathfrak{q}t_\ell}$ and $|\eta| \sim 2^{\mathfrak{q}s_\ell}$. If $t_\ell > s_\ell$, we have $|\xi| \gtrsim |\eta|$. By replacing ℓ with $\imath$ in (6. 4), the estimate in (6. 10) follows. If $t_\ell < s_\ell$, we have $|\xi| \lesssim |\eta|$. By replacing ℓ with $\jmath$ in (6. 4), the estimate in (6. 9) follows.

3. Observe that $|\xi_i-\eta_i| \sim \max\left\{2^{t_i}, 2^{s_i}\right\}$ for $i=\ell,\imath,\jmath$ respectively in (6. 8)-(6. 10). By (3. 27), $\delta_{\mathbf{s}}(\eta)\Big(\Delta_{y_i}\sigma\Big)(y,\eta)/|\xi_i-\eta_i|^2$ satisfies the differential inequality in (2. 1) with its norm bounded by $2^{-(\mathfrak{q}-1)|t_i-s_i|/\mathfrak{q}}$. By carrying out the same estimation inductively on every i whereas $t_i \neq s_i$, and applying Lemma 2.1, we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha} \delta_{\mathbf{t}}(\xi)\Lambda_{\mathbf{s}}(x,\xi)\right| ~\lesssim~ \prod_{i=1}^n 2^{-\left(\frac{\mathfrak{q}-1}{\mathfrak{q}}\right)|t_i-s_i|}\left(\frac{1}{1+|\xi_i|+|\xi|^{\rho}}\right)^{|\alpha^i|} \tag{6. 11}$$

for every multi-index $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

From the same estimates in (5. 4)-(5. 7), with $\Omega_{\mathbf{t}}$ in (5. 2) replaced by $\Omega_{\mathbf{ts}}$ in (6. 2), we have

$$\Big(\Delta_{\mathbf{t}}T_\sigma\Delta_{\mathbf{s}}f\Big)(x) ~\lesssim~ \prod_{i=1}^n 2^{-(1-\rho)|t_i-s_i|} \int f(x-\mathbf{t}z)\left(\frac{1}{1+|z|}\right)^N dz \tag{6. 12}$$

for every $N \geq 1$.

The Lemma is proved by following the same estimates given at (5. 8)-(5. 9). □

7 Conclusion on the Main Result

We conclude the $\mathbf{L}^p$-boundedness of T_σ for $\sigma \in \mathbf{S}_\rho$, in analogue to Carbery and Seeger [10]. Let $f \in \mathbf{L}^2 \cap \mathbf{L}^p$ and $g \in \mathbf{L}^2 \cap \mathbf{L}^q$ for which $1/p + 1/q = 1$. Consider

$$\int \Big(T_\sigma f\Big)(x) g(x) dx \;=\; \int \left\{ \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} T_\sigma f\Big)(x) \right\} \left\{ \sum_{\mathbf{s}} \Big(\Delta_{\mathbf{s}} g\Big)(x) \right\} dx. \tag{7. 1}$$

From (3. 5)-(3. 6), the support of $\delta_{\mathbf{t}}(\xi)\overline{\delta_{\mathbf{s}}(\xi)}$ is nonempty only if $|t_i - s_i| < 2$ for $i = 1, 2, \ldots, n$. By Plancherel theorem, it is suffice to consider $\mathbf{t} = \mathbf{s}$ in (7. 1).

Let h_i be an integer for every $i = 1, 2, \ldots, n$. We define the n-tuples $(\mathbf{t} + \mathbf{h})_i$ and $\mathbf{t} + \mathbf{h}$ by simply replacing t_i with $t_i + h_i$ respectively in (3. 1) and (3. 3). For each $\mathbf{t}$ fixed, we write $f(x) = \sum_{\mathbf{h}} \Big(\Delta_{\mathbf{t}+\mathbf{h}} f\Big)(x)$. Therefore, the right hand side of (7. 1) is now replaced by a constant multiple of

$$\sum_{\mathbf{h}} \left\{ \int \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} T_\sigma \Delta_{\mathbf{t}+\mathbf{h}} f\Big)(x) \Big(\Delta_{\mathbf{t}} g\Big)(x) dx \right\}. \tag{7. 2}$$

It is suffice to assume that all $\mathbf{t}$ and $\mathbf{t}+\mathbf{h}$ in (7. 2) satisfy the hypothesis ($\mathbf{H}$) in (3. 8), whereas the remaining operators $\left(I - \sum_{\mathbf{t}} \Delta_{\mathbf{t}}\right) T_\sigma$ and $T_\sigma \left(I - \sum_{\mathbf{t}+\mathbf{h}} \Delta_{\mathbf{t}+\mathbf{h}}\right)$ have symbols compactly supported in ξ for $0 \leq \rho < 1$. By carrying out the same estimates in section 5, their kernels belong to $C^\infty\left(\mathbb{R}^{\mathbf{N}}\right)$ and decay rapidly at infinity.

By applying Schwarz inequality and then Hölder inequality, we have

$$\begin{aligned} \int \Big(T_\sigma f\Big)(x) g(x) dx \;&\lesssim\; \sum_{\mathbf{h}} \int \left\{ \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} T_\sigma \Delta_{\mathbf{t}+\mathbf{h}} f\Big)^2(x) \right\}^{\frac{1}{2}} \left\{ \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} g\Big)^2(x) \right\}^{\frac{1}{2}} dx \\ &\lesssim\; \sum_{\mathbf{h}} \left\| \left\{ \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} T_\sigma \Delta_{\mathbf{t}+\mathbf{h}} f\Big)^2 \right\}^{\frac{1}{2}} \right\|_{\mathbf{L}^p} \left\| \left\{ \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} g\Big)^2 \right\}^{\frac{1}{2}} \right\|_{\mathbf{L}^q}. \end{aligned} \tag{7. 3}$$

The restriction of the $\mathbf{L}^2$-boundedness can be removed by taking a sequence of functions $f_j \in \mathbf{L}^2 \cap \mathbf{L}^p$ converging to $f \in \mathbf{L}^p$ as $j \longrightarrow \infty$ in $\mathbf{L}^p$, and then using the inequalities in (7. 3).

By taking the supremum of all g with $\|g\|_{\mathbf{L}^q} = 1$ on the left of inequality (7. 3), and using the Littlewood-Paley inequality in (A. 1) on the right, we have

$$\Big\|T_\sigma f\Big\|_{\mathbf{L}^p} \;\lesssim\; \sum_{\mathbf{h}} \left\| \left\{ \sum_{\mathbf{t}} \Big(\Delta_{\mathbf{t}} T_\sigma \Delta_{\mathbf{t}+\mathbf{h}} f\Big)^2 \right\}^{\frac{1}{2}} \right\|_{\mathbf{L}^p}. \tag{7. 4}$$

By Lemma 5.1, $\Delta_{\mathbf{t}} T_\sigma f$ is bounded by $\mathbf{M} f$ for every $\mathbf{t}$ satisfying ($\mathbf{H}$) in (3. 8).

Let $\mathbf{h}$ to be fixed. By using the vector-value inequality of strong maximal function [4], and then the Littlewood-Paley inequality (A. 1), we have

$$\begin{aligned}\left\|\left\{\sum_{\mathbf{t}}\left(\Delta_{\mathbf{t}}T_\sigma\Delta_{\mathbf{t}+\mathbf{h}}f\right)^2\right\}^{\frac{1}{2}}\right\|_{\mathbf{L}^r} &\lesssim \left\|\left\{\sum_{\mathbf{t}}\left(\mathbf{M}\Delta_{\mathbf{t}+\mathbf{h}}f\right)^2\right\}^{\frac{1}{2}}\right\|_{\mathbf{L}^r}\\ &\lesssim \left\|\left\{\sum_{\mathbf{t}}\left(\Delta_{\mathbf{t}+\mathbf{h}}f\right)^2\right\}^{\frac{1}{2}}\right\|_{\mathbf{L}^r} \lesssim \left\|f\right\|_{\mathbf{L}^r}, \qquad 1<r<\infty.\end{aligned} \tag{7. 5}$$

On the other hand, we have $(\Delta_{\mathbf{t}}T_\sigma\Delta_{\mathbf{s}})^* = \Delta^*_{\mathbf{s}}T^*_\sigma\Delta^*_{\mathbf{t}}$. Recall that $\mathbf{supp}\delta_{\mathbf{t}}(\xi)\overline{\delta_{\mathbf{s}}(\xi)} = \emptyset$ provided that $|t_i - s_i| \geq 2$ for some $i = 1, 2, \ldots, n$. By applying Cotlar-Stein Lemma [7] and using Lemma 6.1, for each $\mathbf{h}$ fixed, we have

$$\left\|\left\{\sum_{\mathbf{t}}\left(\Delta_{\mathbf{t}}T_\sigma\Delta_{\mathbf{t}+\mathbf{h}}f\right)^2\right\}^{\frac{1}{2}}\right\|_{\mathbf{L}^2} \lesssim \prod_{i=1}^n 2^{-(1-\rho)|h_i|}\left\|f\right\|_{\mathbf{L}^2}. \tag{7. 6}$$

Let $p \in (r, 2]$ and $p \in [2, r)$. By applying Riesz interpolation theorem [5], from (7. 5)-(7. 6), we have

$$\left\|\left\{\sum_{\mathbf{t}}\left(\Delta_{\mathbf{t}}T_\sigma\Delta_{\mathbf{t}+\mathbf{h}}f\right)^2\right\}^{\frac{1}{2}}\right\|_{\mathbf{L}^p} \lesssim \prod_{i=1}^n 2^{-\varepsilon|h_i|}\left\|f\right\|_{\mathbf{L}^p} \tag{7. 7}$$

for some $\varepsilon = \varepsilon(\rho, p) > 0$. By summing over all the h_i s in (7. 4), we obtain the desired result.

A Littlewood-Paley Inequality

In this appendix, we prove the Littlewood-Paley inequality applied in section 7. Namely

$$\left\|\left\{\sum_{\mathbf{t}}\left(\Delta_{\mathbf{t}}f\right)^2\right\}^{\frac{1}{2}}\right\|_{\mathbf{L}^p} \lesssim \left\|f\right\|_{\mathbf{L}^p}, \qquad 1<p<\infty. \tag{A. 1}$$

Notice that for $\rho = 0$, the inequality in (A. 1) is proved by R.Fefferman and Stein [1]. It is suffice to take $n = 2$ and write $(x, y) \in \mathbb{R}^{\mathbf{N}_1} \times \mathbb{R}^{\mathbf{N}_2}$. Let (ξ, η) to be the dual variables of (x, y) in the frequency space. Recall the function ϕ given in (3. 5). Let δ be a positive real number. For $0 < \rho < 1$, we define

$$\widehat{\Phi}_\delta(\xi, \eta) = \phi\left(\delta\xi, \delta^{\frac{1}{\rho}}\eta\right). \tag{A. 2}$$

By definition ϕ in (3. 5), we have

$$\iint \Phi_\delta(x, y)dxdy = 0 \tag{A. 3}$$

for every $\delta > 0$. Moreover, $\Phi = \Phi_1$ is smooth, bounded and decays rapidly as $|(x, y)| \longrightarrow \infty$. The square function operator $\mathfrak{S}_\Phi$ is defined by

$$\left(\mathfrak{S}_\Phi f\right)(x, y) = \left\{\int_0^\infty \left(f * \Phi_\delta(x, y)\right)^2 \frac{d\delta}{\delta}\right\}^{\frac{1}{2}}. \tag{A. 4}$$

Its kernel, denoted by Ω, is a Hilbert space-value function such that

$$\Omega(x,y) \;=\; \Phi_\delta(x,y) \;=\; \frac{1}{\delta^{\mathbf{N}_1}\delta^{\mathbf{N}_2/\rho}}\Phi\left(\frac{x}{\delta},\frac{y}{\delta^{1/\rho}}\right) \tag{A. 5}$$

with its norm defined by

$$\left|\Omega(x,y)\right|_H \;=\; \left\{\int_0^\infty \left|\Phi_\delta(x,y)\right|^2 \frac{d\delta}{\delta}\right\}^{\frac{1}{2}}. \tag{A. 6}$$

Lemma A.1

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta \Omega(x,y)\right|_H \;\leq\; A_{\alpha\beta}\left(\frac{1}{|x|+|y|^\rho}\right)^{\mathbf{N}_1+|\alpha|}\left(\frac{1}{|y|+|x|^{1/\rho}}\right)^{\mathbf{N}_2+|\beta|} \tag{A. 7}$$

for every multi-index α *and* β *away from the origin.*

Proof: 1. Suppose $|x|\leq\delta$ and $|y|\leq\delta^{1/\rho}$. Since Φ is smooth and bounded, (A. 5) implies

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left\{\int_{|x|}^\infty \left(\frac{1}{|x|}\right)^{2\mathbf{N}_1+2\mathbf{N}_2/\rho-1}\frac{d\delta}{\delta^2}\right\}^{\frac{1}{2}} \;\sim\; \left(\frac{1}{|x|}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho} \tag{A. 8}$$

and

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left\{\int_{|y|^\rho}^\infty \left(\frac{1}{|y|}\right)^{2\rho\mathbf{N}_1+2\mathbf{N}_2-1}\frac{d\delta}{\delta^{1+1/\rho}}\right\}^{\frac{1}{2}} \;\sim\; \left(\frac{1}{|y|}\right)^{\rho\mathbf{N}_1+\mathbf{N}_2}. \tag{A. 9}$$

2. Suppose $|x|>\delta$ and $|y|\leq\delta^{1/\rho}$. We necessarily have $|x|>|y|^\rho$. Since Φ decays rapidly as $|(x,y)|\longrightarrow\infty$, we have

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left\{\int_0^{|x|} \left(\frac{1}{|x|}\right)^{2\mathbf{N}_1+2\mathbf{N}_2/\rho+2}\delta d\delta\right\}^{\frac{1}{2}} \;\sim\; \left(\frac{1}{|x|}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho}. \tag{A. 10}$$

On the other hand, the estimate in (A. 9) implies

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left(\frac{1}{|y|}\right)^{\rho\mathbf{N}_1+\mathbf{N}_2}. \tag{A. 11}$$

3. Suppose $|x|\leq\delta$ and $|y|>\delta^{1/\rho}$. We necessarily have $|x|<|y|^\rho$. The estimate in (A. 8) implies

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left(\frac{1}{|x|}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho}. \tag{A. 12}$$

On the other hand, since Φ decays rapidly as $|(x,y)|\longrightarrow\infty$, we have

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left\{\int_0^{|y|^\rho} \left(\frac{1}{|y|}\right)^{2\rho\mathbf{N}_1+2\mathbf{N}_2+2}\delta^{2/\rho-1}d\delta\right\}^{\frac{1}{2}} \;\sim\; \left(\frac{1}{|y|}\right)^{\rho\mathbf{N}_1+\mathbf{N}_2}. \tag{A. 13}$$

4. Suppose $|x|>\delta$ and $|y|>\delta^{1/\rho}$. By the estimates in (A. 10) and (A. 13), we have

$$\left|\Omega(x,y)\right|_H \;\lesssim\; \left(\frac{1}{|x|}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho}, \qquad \left|\Omega(x,y)\right|_H \;\lesssim\; \left(\frac{1}{|y|}\right)^{\rho\mathbf{N}_1+\mathbf{N}_2}. \tag{A. 14}$$

All together, we obtain

$$\Big|\Omega(x,y)\Big|_H \lesssim \left(\frac{1}{|x|+|y|^\rho}\right)^{\mathbf{N}_1}\left(\frac{1}{|y|+|x|^{1/\rho}}\right)^{\mathbf{N}_2}. \tag{A. 15}$$

Observe that every ∂_x acting on Φ_δ gains a constant multiple of δ^{-1} and every ∂_y acting on Φ_δ gains a constant multiple of $\delta^{-1/\rho}$ respectively. By carrying out the same estimates as above, we prove the differential inequality in (A. 7). □

Define the distance function

$$\rho(x,y) = \max\Big\{|x|,|y|^\rho\Big\} \tag{A. 16}$$

and the nonisotropic ball

$$\mathbf{B}_{\rho,\delta} = \Big\{(x,y)\in\mathbb{R}^{\mathbf{N}_1}\times\mathbb{R}^{\mathbf{N}_2} : \rho(x,y)\le\delta\Big\}. \tag{A. 17}$$

Let $(u,v)\in\mathbf{B}_{\rho,\delta}$ and $(x,y)\in{}^c\mathbf{B}_{\rho,r\delta}$ for some $r>1$. By adjusting its value, we can have

$$\rho(x,y) > r\delta, \qquad \rho(x-u,y-v) > \delta. \tag{A. 18}$$

The differential inequality in (A. 7) implies

$$\begin{aligned}
&\Big|\Omega(x-u,y-v)-\Omega(x,y)\Big|_H \\
&\lesssim \frac{|u|}{\rho(x,y)}\left(\frac{1}{\rho(x,y)}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho} + \frac{|v|}{\rho(x,y)^{1/\rho}}\left(\frac{1}{\rho(x,y)}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho} \\
&\lesssim \frac{\rho(u,v)}{\rho(x,y)}\left(\frac{1}{\rho(x,y)}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho}.
\end{aligned} \tag{A. 19}$$

From (A. 18) and (A. 19), we have

$$\begin{aligned}
&\sum_{k=0}^\infty \iint_{\mathbf{B}_{\rho,2^{k+1}r\delta}\setminus\mathbf{B}_{\rho,2^k r\delta}} \Big|\Omega(x-u,y-v)-\Omega(x,y)\Big|_H\,dxdy \\
&\lesssim \sum_{k=0}^\infty \iint_{\mathbf{B}_{\rho,2^{k+1}r\delta}\setminus\mathbf{B}_{\rho,2^k r\delta}} \frac{\rho(u,v)}{\rho(x,y)}\left(\frac{1}{\rho(x,y)}\right)^{\mathbf{N}_1+\mathbf{N}_2/\rho} dxdy \\
&\lesssim 2^{\mathbf{N}_1+\mathbf{N}_2/\rho}\sum_{k=0}^\infty 2^{-k}, \qquad (u,v)\in\mathbf{B}_{\rho,\delta}.
\end{aligned} \tag{A. 20}$$

By (A. 20), we have

$$\iint_{{}^c\mathbf{B}_{\rho,r\delta}} \Big|\Omega(x-u,y-v)-\Omega(x,y)\Big|_H\,dxdy \le A \tag{A. 21}$$

for $(u,v)\in\mathbf{B}_{\rho,\delta}$.

This is the well known condition as discussed in chapter I of [7] for which $f*\Omega$ satisfies the weak type $(1,1)$-estimate as a Hilbert space-value function in (A. 5)-(A. 6).

On the other hand, by Plancherel theorem, we have

$$\begin{aligned}\left\|\mathfrak{S}_{\Phi}f\right\|_{\mathbf{L}^2}^2 &= \iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \left\{\int_0^\infty |\widehat{f}(\xi,\eta)|^2 \left|\widehat{\Phi}\left(\delta\xi,\delta^{\frac{1}{\rho}}\eta\right)\right|^2 \frac{d\delta}{\delta}\right\} d\xi d\eta \\ &\leq \left\{\sup_{(\xi,\eta)} \int_0^\infty \phi^2\left(\delta\xi,\delta^{\frac{1}{\rho}}\eta\right)\frac{d\delta}{\delta}\right\} \iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} |\widehat{f}(\xi,\eta)|^2 d\xi d\eta \lesssim \left\|f\right\|_{\mathbf{L}^2}^2\end{aligned} \tag{A. 22}$$

where the second inequality is followed by the properties of ϕ defined in (3. 5).

Let λ to be a positive real number and define $\widehat{\Psi}_\lambda(\xi,\eta) = \phi(\lambda^{\frac{1}{\rho}}\xi, \lambda\eta)$. Our estimates proving Lemma 7.1 remain to be valid for $\Omega = \Psi_\lambda$ as in (A. 5)-(A. 6), with x and y switched in roles. Let $F = f * \Phi_\delta$. By applying the $\mathbf{L}^p$-regularity theorem for space-value functions given in chapter I of [7] , we have

$$\begin{aligned}\iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \left(\mathfrak{S}_{\Phi*\Psi}f\right)^p(x,y)dxdy &= \iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \left\{\int_0^\infty \left|\left(F*\Psi_\lambda\right)(x,y)\right|_H^2 \frac{d\lambda}{\lambda}\right\}^{\frac{p}{2}} dxdy \\ &\lesssim \iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \left|F(x,y)\right|_H^p dxdy \\ &= \iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \left\{\int_0^\infty \left|\left(f*\Phi_\delta\right)(x,y)\right|^2 \frac{d\delta}{\delta}\right\}^{\frac{p}{2}} dxdy \\ &\lesssim \iint_{\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}} \left|f(x,y)\right|^p dxdy.\end{aligned} \tag{A. 23}$$

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