WEIGHTED INEQUALITIES FOR PRODUCT FRACTIONAL INTEGRALS

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Abstract. We investigate one and two weight norm inequalities for product fractional integrals

$$I_{\alpha,\beta}^{m,n} f(x,y) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |x-u|^{\alpha-m} |y-t|^{\beta-n} f(u,t)\, du dt$$

in $\mathbb{R}^m \times \mathbb{R}^n$. We show that in the one weight case, most of the 1-parameter theory carries over to the 2-parameter setting - the one weight inequality

$$\left\| \left(I_{\alpha,\beta}^{m,n} f \right) w \right\|_{L^q} \leq N_{p,q}^{(\alpha,m),(\beta,n)} \|fw\|_{L^p}, \qquad f \geq 0,$$

is equivalent to finiteness of the rectangle characteristic

$$A_{p,q}^{(\alpha,\beta),(m,n)}(w) = \sup_{I,J} |I|^{\frac{\alpha}{m}-1} |J|^{\frac{\beta}{n}-1} \left(\iint_{I\times J} w^q \right)^{\frac{1}{q}} \left(\iint_{I\times J} w^{-p'} \right)^{\frac{1}{p'}},$$

which is in turn equivalent to the diagonal and balanced equalities

$$0 < \frac{\alpha}{m} = \frac{\beta}{n} = \frac{1}{p} - \frac{1}{q}.$$

Moreover, the optimal power of the characteristic that bounds the norm is $2 + 2\max\left\{ \frac{p'}{q}, \frac{q}{p'} \right\}$.

However, in the two weight case, apart from the trivial case of product weights, we show that virtually *none* of the standard results carry over from the 1-parameter setting - the two-tailed characteristic

$$\widehat{A}_{p,q}^{(\alpha,\beta),(m,n)}(v,w) = \sup_{I,J} |I|^{\frac{\alpha}{m}-1} |J|^{\frac{\beta}{n}-1}$$
$$\times \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left(\widehat{s}_{I\times J} w\right)^q d\omega \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left(\widehat{s}_{I\times J} v^{-1}\right)^{p'} d\sigma \right)^{\frac{1}{p'}}$$

fails to control the operator norm of $I_{\alpha,\beta} : L^p(v^p) \to L^q(w^q)$ in general, and even when both weights $v^{-p'}$ and w^q satisfy the rectangle A_1 condition in the case of strictly subbalanced diagonal indices $0 < \frac{1}{p} - \frac{1}{q} < \frac{\alpha}{m} = \frac{\beta}{n}$. And finally, not even the rectangle testing conditions are sufficient for the two weight inequality when both weights satisfy the rectangle A_1 condition. Nevertheless, the Stein-Weiss extension of the classical Hardy - Littlewood - Sobolev inequality to power weights carries over to the 2-parameter setting with *nonproduct* power weights using a 'sandwiching' technique, providing our main positive result in two weight 2-parameter theory.

Contents

1. Introduction

The theory of weighted norm inequalities for product operators, i.e. those operators commuting with a multiparameter family of dilations, has seen little progress since the pioneering work of Robert Fefferman [Fef] in the 1980's involving covering lemmas for collections of rectangles. The purpose of the present paper is to settle some of the basic questions arising in the weighted theory for the special case of product *fractional* integrals, in particular the relationship between their norm inequalities and their associated characteristics, two-tailed characteristics, and testing conditions. Our four main results can be split into two distinct parts, which are presented largely independent of each other:

(1) In the first part of the paper, we consider the general two weight norm inequality $I^{m,n}_{\alpha,\beta} : L^p(v^p) \to L^q(w^q)$ for **product** fractional integrals $I^{m,n}_{\alpha,\beta}$ when the indices are *subbalanced*:

$$\min\left\{\frac{\alpha}{m}, \frac{\beta}{n}\right\} \geq \frac{1}{p} - \frac{1}{q} > 0.$$

We focus separately on the three subcases where the indices are *balanced* $\frac{\alpha}{m} = \frac{\beta}{n} = \frac{1}{p} - \frac{1}{q} > 0$, *half subbalanced* $\min\left\{\frac{\alpha}{m}, \frac{\beta}{n}\right\} = \frac{1}{p} - \frac{1}{q} > 0$ with $\frac{\alpha}{m} \neq \frac{\beta}{n}$, and finally *strictly subbalanced*, $\min\left\{\frac{\alpha}{m}, \frac{\beta}{n}\right\} > \frac{1}{p} - \frac{1}{q} > 0$. It turns out that the one weight theory in 1-parameter carries over to the product setting in the balanced case, some of the familiar two weight theory in 1-parameter carries over in the half balanced case, and finally, *none* of the familiar two weight theory in 1-parameter carries over in the strictly subbalanced case. More precisely, we prove:

  (a) In the balanced case, the one weight inequality in the product setting is equivalent to finiteness of the product Muckenhoupt characteristic.
  (b) In the half balanced case, we use the iteration theory and Stein's interpolation of analytic families of operators to show that the two weight inequality holds if in addition to the Muckenhoupt characteristic, we have one of the side conditions, $v^p \in A_p \times A_p$ or $w^q \in A_q \times A_q$.
  (c) In the strictly sub-balanced case, we give a construction of a family of counterexamples to two weight inequalities for product fractional integrals, spoiling any chance that conditions based on the familiar characteristics and $T1$ type testing conditions - that play such an important role in characterizing two weight theory in 1-parameter - can play an analogous role in the multiparameter setting, and perhaps helping to explain the lack of progress in this area since the 1980's.

(2) In the second part of the paper, we give a sharp product version of the Stein-Weiss extension of the classical Hardy-Littlewood-Sobolev theorem for (nonproduct) power weights, which we establish by a (somewhat complicated) method of iteration, something traditionally thought unlikely.

The family of counterexamples in two weight theory requires a delicate construction of rectangle $A_1 \times A_1$ weights with 'nodes' carefully arranged on a hyperbola. Our positive results for one weight theory and two power weight theory are obtained using the tools of iteration, Minkowski's inequality and a sandwiching argument. In fact, it seems to be the case that all of the positive results in weighted product theory to date can be obtained using these three tools. Now we begin to describe these matters in detail.

Let $m,n \geq 1$. For indices $1 < p,q < \infty$ and $0 < \alpha < m$ and $0 < \beta < n$, we consider the weighted norm inequality

$$\left\{ \int\int_{\mathbb{R}^m \times \mathbb{R}^n} I_{\alpha,\beta}^{m,n} f(x,y)^q \ w(x,y)^q \ dxdy \right\}^{\frac{1}{q}} \leq N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \left\{ \int\int_{\mathbb{R}^m \times \mathbb{R}^n} f(u,t)^p \ v(u,t)^p \ dudt \right\}^{\frac{1}{p}}$$

for the product fractional integral

$$I_{\alpha,\beta}^{m,n} f(x,y) = \int\int_{\mathbb{R}^m \times \mathbb{R}^n} |x-u|^{\alpha-m} |y-t|^{\beta-n} f(u,t)\, dudt, \qquad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n,$$

and characterize when one weight and two weight inequalities are equivalent to the corresponding product fractional Muckenhoupt characteristic,

$$A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \equiv \sup_{I \subset \mathbb{R}^m,\ J \subset \mathbb{R}^n} |I|^{\frac{\alpha}{m}-1} |J|^{\frac{\beta}{n}-1} \left(\int\int_{I\times J} w^q \right)^{\frac{1}{q}} \left(\int\int_{I \times J} v^{-p'} \right)^{\frac{1}{p'}}, \tag{1.1}$$

and its two-tailed variant,

$$\begin{aligned} &\widehat{A}_{p,q}^{(m,n),(\alpha,\beta)}(v,w) \\ &\equiv \sup_{I\times J \subset \mathbb{R}^m \times \mathbb{R}^n} |I|^{\frac{\alpha}{m}-1} |J|^{\frac{\beta}{n}-1} \left(\int\int_{\mathbb{R}^m \times \mathbb{R}^n} (\widehat{s}_{I\times J} w)^q \, d\omega \right)^{\frac{1}{q}} \left(\int\int_{\mathbb{R}^m \times \mathbb{R}^n} \left(\widehat{s}_{I\times J} v^{-1}\right)^{p'} d\sigma \right)^{\frac{1}{p'}}, \end{aligned} \tag{1.2}$$

where

$$\widehat{s}_{I\times J}(x,y) \equiv \left(1 + \frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)^{\alpha-m} \left(1 + \frac{|y-c_J|}{|J|^{\frac{1}{n}}}\right)^{\beta-n}. \tag{1.3}$$

For the *one weight* inequality when $v = w$, we show that the indices must be balanced and diagonal, i.e.

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n},$$

and that $p < q$; then the finiteness of the operator norm $N_{p,q}^{(m,n)}(w)$ is equivalent to finiteness of the characteristic $A_{p,q}^{(m,n)}(w)$, where as is conventional, we are suppressing redundant indices in the one weight diagonal balanced case. In addition, we characterize the optimal power of the characteristic that controls the operator norm. These one weight results are proved by an iteration strategy using Minkowski's inequality that also yields two weight results for the special case of *product weights*.

For the general two weight case, we show that in the absence of any side conditions on the weight pair (v,w), the operator norm $N_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ is *never* controlled by the two-tailed characteristic $\widehat{A}_{p,q}^{(m,n),(\alpha,\beta)}(v,w)$, not even the weak type operator norm of the much smaller dyadic fractional maximal function $M_{\alpha,\beta}^{\mathrm{dy}}$ (see below for definitions). On the other hand, new two weight results can be obtained from known norm inequalities, such as the one weight and product weight results mentioned above, by the technique of 'sandwiching'.

Lemma 1. *If $\left\{\left(V^i, W^i\right)\right\}_{i=1}^N$ is a sequence of weight pairs 'sandwiched' in a weight pair (v,w), i.e.*

$$\frac{w(x,y)}{v(u,t)} \leq \sum_{i=1}^N \frac{W^i(x,y)}{V^i(u,t)},$$

then[1]

$$N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \leq \sum_{i=1}^{N} N_{p,q}^{(\alpha,\beta),(m,n)}\left(V^i, W^i\right).$$

Proof. This follows immediately from setting $g = fV$ in the identity,

$$N_{p,q}^{(\alpha,\beta),(m,n)}(V,W) = \sup_{\|g\|_{L^p}\leq 1, \|h\|_{L^{q'}}\leq 1} \int_{\mathbb{R}^m}\int_{\times\mathbb{R}^n} h(x,y)\, W(x,y)\, |x-u|^{\alpha-m}\, |y-t|^{\beta-n}\, \frac{g(u,t)}{V(u,t)} dxdydudt.$$

□

We mention two simple examples of sandwiching, the first example sandwiching a one weight pair, and the second example sandwiching two product weight pairs:

(1) In the case of diagonal and balanced indices $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n}$, if there is a *one weight* pair (u,u) with $A_{p,q}^{(\alpha,\beta),(m,n)}(u) < \infty$ sandwiched in (v,w), then $N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \lesssim A_{p,q}^{(\alpha,\beta),(m,n)}(u) < \infty$.

(2) If $w(x,y) = |(x,y)|^{-\gamma}$ and $v(x,y) = |(x,y)|^{\delta}$ are power weights on $\mathbb{R}^{m+n}$, then $N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) < \infty$ provided the indices satisfy product conditions corresponding to those of the Stein-Weiss theorem in 1-parameter (see below), that in turn generalize the classical Hardy-Littlewood-Sobolev inequality. In this example there are two weight pairs $\{(V,W),(V',W')\}$ depending on the indices and sandwiched in (v,w) where each weight is an appropriate product power weight.

On the other hand, we exhibit examples that show that even if *both* weights satisfy the product $A_1 \times A_1$ condition, the operator norm $N_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ still cannot be controlled by $\widehat{A}_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ when $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{m} = \frac{\beta}{n}$. Moreover, under these conditions, not even the testing conditions, where the norm inequality and its dual are tested over rectangles times w^q and $v^{-p'}$ respectively, are sufficient for the weighted norm inequality to hold.

Problem 1. *A basic open problem remains - that of finding a new type of condition to characterize the two weight norm inequality for product fractional integrals, given that all of the traditional candidates fail. For example, testing over all indicators of sets is not yet ruled out.*

Thus we see that the two weight product situation turns out to have striking differences from that of the 1-parameter theory, and to highlight this, we begin by briefly recalling the 1-parameter weighted theory of fractional integrals.

2. 1-parameter theory

Define $\Omega_\alpha^m(x) = |x|^{\alpha-m}$ and set $I_\alpha^m g = \Omega_\alpha^m * g$. The following one weight theorem for fractional integrals is due to Muckenhoupt and Wheeden.

Theorem 1. *Let* $0 < \alpha < m$. *Suppose* $1 < p < q < \infty$ *and*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m}. \tag{2.1}$$

Let $w(x)$ *be a nonnegative weight on* $\mathbb{R}^m$. *Then*

$$\left\{\int_{\mathbb{R}^m} I_\alpha^m f(x)^q\ \ w(x)^q\ \ dx\right\}^{\frac{1}{q}} \leq N_{p,q}(w) \left\{\int_{\mathbb{R}^m} f(x)^p\ \ w(x)^p\ \ dx\right\}^{\frac{1}{p}}$$

for all $f \geq 0$ *for* $N_{p,q}(w) < \infty$ if and only if

$$A_{p,q}(w) \equiv \sup_{cubes\ I\subset\mathbb{R}^m} \left(\frac{1}{|I|}\int_I w(x)^q\ \ dx\right)^{\frac{1}{q}} \left(\frac{1}{|I|}\int_I w(x)^{-p'}\ \ dx\right)^{\frac{1}{p'}} < \infty. \tag{2.2}$$

[1] In the case $N = 1$, the hypothesis is implied by $w \leq W^1$ and $V^1 \leq v$, hence the terminology 'sandwiched'.

In fact, assuming only that $1<p,q<\infty$, the balanced condition (2.1) is *necessary* for the norm inequality (2.2). See Theorem 14 in the Appendix for this.

A two weight analogue for $1<p\leq q<\infty$ was later obtained by Sawyer [Saw] that involved testing the norm inequality and its dual over indicators of cubes times w^q and $v^{-p'}$ respectively, namely for all cubes $Q\subset\mathbb{R}^m$,

$$\begin{aligned}\left\{\int_{\mathbb{R}^m} I_\alpha^m\left(\mathbf{1}_Q v^{-p'}\right)(x)^q\ \ w(x)^q\ \ dx\right\}^{\frac{1}{q}} &\leq T_{p,q}^{\alpha,m}(v,w)\ \ |Q|_{v^{-p'}}^{\frac{1}{p}},\\ \left\{\int_{\mathbb{R}^m} I_\alpha^m\left(\mathbf{1}_Q w^q\right)(x)^{p'}\ \ v(x)^{-p'}\ \ dx\right\}^{\frac{1}{p'}} &\leq T_{q',p'}^{\alpha,m}\left(\frac{1}{w},\frac{1}{v}\right)\ \ |Q|_{w^q}^{\frac{1}{q'}}.\end{aligned}$$

Later yet, it was shown by Sawyer and Wheeden [SaWh], using an idea of Kokilashvili and Gabidzashvili [KoGa], that in the special case $p<q$, the testing conditions could be replaced with a two-tailed two weight version of the $A_{p,q}$ condition (2.2):

$$\sup_{I\subset\mathbb{R}^m}|I|^{\frac{\alpha}{m}-1}\left(\frac{1}{|I|}\int_I\left[\widehat{s}_I(x)\,w(x)\right]^q\ \ dx\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I\left[\widehat{s}_I(x)\,v(x)^{-1}\right]^{p'}\ \ dx\right)^{\frac{1}{p'}}\equiv\widehat{A}_{p,q}^{\alpha,m}(v,w)<\infty, \tag{2.3}$$

where the tail $\widehat{s}_I$ is given by

$$\widehat{s}_I(x)\equiv\left(1+\frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)^{\alpha-m},$$

and c_I is the center of the cube I. Note that in Sawyer and Wheeden [SaWh], this condition was written in terms of the rescaled tail $s_I=|I|^{\frac{\alpha}{m}-1}\widehat{s}_I$, and it was also shown that the two-tailed condition $\widehat{A}_{p,q}$ condition (2.3) could be replaced by a pair of corresponding one-tailed conditions.

Theorem 2. *Suppose $1<p<q<\infty$ and $0<\alpha<m$. Let $w(x)$ and $v(x)$ be a pair of nonnegative weights on $\mathbb{R}^m$. Then*

$$\left\{\int_{\mathbb{R}^m} I_\alpha^m f(x)^q\ \ w(x)^q\ \ dx\right\}^{\frac{1}{q}}\leq N_{p,q}^{\alpha,m}(v,w)\left\{\int_{\mathbb{R}^m} f(x)^p\ \ v(x)^p\ \ dx\right\}^{\frac{1}{p}} \tag{2.4}$$

for all $f\geq 0$ if and only if *the $\widehat{A}_{p,q}^{\alpha,m}$ condition (2.3) holds, i.e. $\widehat{A}_{p,q}^{\alpha,m}(v,w)<\infty$. Moreover, the best constant $N_{p,q}^{\alpha,m}(w,v)$ in (2.4) is comparable to $\widehat{A}_{p,q}^{\alpha,m}(w,v)$.*

The special case of power weights $|x|^\gamma$ had been considered much earlier, and culminated in the following 1958 theorem of Stein and Weiss [StWe2].

Theorem 3. *Let $w_\gamma(x)=|x|^{-\gamma}$ and $v_\delta(x)=|x|^\delta$ be a pair of nonnegative power weights on $\mathbb{R}^m$ with $-\infty<\gamma,\delta<\infty$. Suppose $1<p\leq q<\infty$ and $\alpha\in\mathbb{R}$ satisfy the strict constraint inequalities,*

$$0<\alpha<m \text{ and } q\gamma<m \text{ and } p'\delta<m, \tag{2.5}$$

together with the inequality

$$\gamma+\delta\geq 0,$$

and the power weight equality,

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha-(\gamma+\delta)}{m}.$$

Then (2.4) holds, i.e.

$$\left\{\int_{\mathbb{R}^m} I_\alpha^m f(x)^q\ \ |x|^{-\gamma q}\ \ dx\right\}^{\frac{1}{q}}\leq N_{p,q}^{\alpha,m}(w_\gamma,v_\delta)\left\{\int_{\mathbb{R}^m} f(x)^p\ \ |x|^{\delta p}\ \ dx\right\}^{\frac{1}{p}}.$$

The previous theorem cannot be improved when the weights are restricted to power weights. Indeed, $\alpha>0$ follows from the local integrability of the kernel, both $q\gamma<m$ and $p'\delta<m$ follow from the local integrability of w^q and $v^{-p'}$, and then both $\gamma+\delta\geq 0$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha-(\gamma+\delta)}{m}$ follow from the finiteness of

the Muckenhoupt condition $A_{p,q}(v,w)$ using standard arguments (see e.g. the proof of Theorem 9 below). These conditions then yield

$$\alpha=m\left(\frac{1}{p}-\frac{1}{q}\right)+\gamma+\delta<m\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{m}{q}+\frac{m}{p'}=m.$$

A routine calculation shows that the aforementioned conditions on the indices $\alpha,m,\gamma,\delta,p,q$ are precisely those for which the characteristic $A_{p,q}^{\alpha,m}(w_\gamma,v_\delta)$ is finite. Finally, the necessity of the remaining condition $p\leq q$ is an easy consequence of Maz'ja's characterization [Maz] of the Hardy inequality for $q<p$. See Theorem 15 in the Appendix for this. Altogether, this establishes the succinct conclusion that the power weight norm inequality holds if and only if $p\leq q$ and the characteristic is finite.

Next, in the one weight setting, we recall the solution to the 'power of the characteristic' problem for fractional integrals due to Lacey, Moen, Perez and Torres in [LaMoPeTo]. See the Appendix for a different proof of this 1-parameter theorem that reveals the origin of the number $1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}$ to be the optimal exponent in the inequality $\overline{A}_{p,q}(w)\leq A_{p,q}(w)^{1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}}$ where $\overline{A}_{p,q}(w)$ is a one-tailed version of $A_{p,q}(w)$.

Theorem 4. *Let $0<\alpha<m$. Suppose $1<p\leq q<\infty$ and*

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{m}.$$

Let $w(x)$ be a nonnegative weight on $\mathbb{R}^m$. Then

$$\left\{\int_{\mathbb{R}^m}I_\alpha^m f(x)^q\ w(x)^q\ dx\right\}^{\frac{1}{q}}\leq N_{p,q}(w)\left\{\int_{\mathbb{R}^m}f(x)^p\ w(x)^p\ dx\right\}^{\frac{1}{p}}$$

where

$$N_{p,q}(w)\leq C_{p,q,n}\ A_{p,q}(w)^{1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}}.$$

The power $1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}$ is sharp, even when w is restricted to power weights $w(x)=|x|^\gamma$.

Finally we recall the extremely simple proof of the equivalence of the dyadic characteristic

$$\mathbb{A}_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)\equiv\sup_{Q\subset\mathbb{R}^n\ \mathrm{dyadic}}|Q|^{\frac{\alpha}{m}+\frac{1}{q}-\frac{1}{p}}\left(\frac{|Q|_\omega}{|Q|}\right)^{\frac{1}{q}}\left(\frac{|Q|_\sigma}{|Q|}\right)^{\frac{1}{p'}},$$

where the supremum is taken over dyadic cubes, and the weak type (p,q) operator norm $\mathbb{N}_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)$ of the dyadic fractional maximal operator M_α^{dy} with respect to (σ,ω):

$$\mathbb{N}_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)\equiv\sup_{f\geq 0}\frac{\sup_{\lambda>0}\lambda\left|\left\{M_\alpha^{\mathrm{dy}}(f\sigma)>\lambda\right\}\right|_\omega^{\frac{1}{q}}}{\left(\int f^p d\sigma\right)^{\frac{1}{p}}}.$$

This proof illustrates the power of an effective covering lemma for weights, something that is sorely lacking in the 2-parameter setting, and accounts for much of the negative nature of our results (c.f. Example 1 below). Recall that the 1-parameter dyadic fractional maximal operator M_α^{dy} acts on a signed measure μ in $\mathbb{R}^n$ by

$$M_\alpha^{\mathrm{dy}}\mu(x)\equiv\sup_{Q\ \mathrm{dyadic}}|Q|^{\alpha-n}\int_Q d|\mu|\ .$$

Lemma 2. *Let (σ,ω) be a locally finite weight pair in $\mathbb{R}^n$, and let $1<p\leq q<\infty$. Then $M_\alpha^{\mathrm{dy}}:L^p(\sigma)\to L^{q,\infty}(\omega)$ if and only if $A_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)<\infty$.*

Proof. Fix $\lambda>0$ and $f\geq 0$ bounded with compact support. Let $\Omega_\lambda\equiv\left\{x\in\mathbb{R}^n:M_\alpha^{\mathrm{dy}}f\sigma>\lambda\right\}$. Then

$$\Omega_\lambda=\dot{\bigcup_k}Q_k\ ,\qquad|Q_k|^{\alpha-n}\int_{Q_k}fd\sigma>\lambda,$$

and we have

$$\begin{aligned}\left(\lambda\left|\Omega_\lambda\right|_\omega^{\frac{1}{q}}\right)^p &= \lambda^p\left(\sum_k\left|Q_k\right|_\omega\right)^{\frac{p}{q}} \leq \lambda^p\sum_k\left|Q_k\right|_\omega^{\frac{p}{q}} = \sum_k\lambda^p\left|Q_k\right|_\omega^{\frac{p}{q}}\\ &< \sum_k\left(\left|Q_k\right|^{\alpha-n}\int_{Q_k}fd\sigma\right)^p\left|Q_k\right|_\omega^{\frac{p}{q}} = \sum_k\left(\left|Q_k\right|^{\alpha-n}\left|Q_k\right|_\sigma^{\frac{1}{p'}}\left|Q_k\right|_\omega^{\frac{1}{q}}\right)^p\left|Q_k\right|_\sigma^{1-p}\int_{Q_k}fd\sigma\\ &\leq \sum_k\left(A_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)\right)^p\int_{Q_k}f^pd\sigma \leq A_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)^p\left\|f\right\|_{L^p(\sigma)}^p ,\end{aligned}$$

which gives

$$\begin{aligned}\left\|M_\alpha^{\mathrm{dy}}f\right\|_{L^{q,\infty}(\omega)} &= \sup_{\lambda>0}\lambda\left|\Omega_\lambda\right|_\omega^{\frac{1}{q}} \leq A_{p,q}^{\alpha,\mathrm{dy}}(\sigma,\omega)\left\|f\right\|_{L^p(\sigma)} ,\\ \text{for all } f &\geq 0 \text{ bounded with compact support.}\end{aligned}$$

The proof of the converse statement is standard, similar to but easier than that of Lemma 3 below. □

3. 2-parameter theory

Define the product fractional integral $I_{\alpha,\beta}^{m,n}$ on $\mathbb{R}^m\times\mathbb{R}^n$ by the convolution formula $I_{\alpha,\beta}^{m,n}f\equiv\Omega_{\alpha,\beta}^{m,n}*f$, where the convolution kernel $\Omega_{\alpha,\beta}^{m,n}$ is a product function:

$$\Omega_{\alpha,\beta}^{m,n}(x,y)=\left|x\right|^{\alpha-m}\left|y\right|^{\beta-n},\qquad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n.$$

Let $v(x,y)$ and $w(x,y)$ be positive weights on $\mathbb{R}^m\times\mathbb{R}^n$. For $1<p,q<\infty$ and $0<\alpha<m$, $0<\beta<n$, we consider the two weight norm inequality for nonnegative functions $f(x,y)$:

(3.1)
$$\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}I_{\alpha,\beta}^{m,n}f(x,y)^q\ \ w(x,y)^q\ \ dxdy\right\}^{\frac{1}{q}}\leq N_{p,q}^{(\alpha,\beta),(m,n)}(v,w)\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}f(x,y)^p\ \ v(x,y)^p\ \ dxdy\right\}^{\frac{1}{p}}.$$

If we define absolutely continuous measures σ,ω by

$$d\sigma(x,y)=v(x,y)^{-p'}dxdy \text{ and } d\omega(x,y)=w(x,y)^q dxdy, \tag{3.2}$$

then the two weight norm inequality (3.1) is equivalent to the norm inequality

$$\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}I_{\alpha,\beta}^{m,n}(f\sigma)(x,y)^q\ \ d\omega(x,y)\right\}^{\frac{1}{q}}\leq\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}f(x,y)^p\ \ d\sigma(x,y)\right\}^{\frac{1}{p}}, \tag{3.3}$$

where now the measure σ appears inside the argument of $I_{\alpha,\beta}^{m,n}$, namely in $I_{\alpha,\beta}^{m,n}(f\sigma)$. In this form, the norm inequality makes sense for arbitrary locally finite Borel measures σ,ω since for nonnegative $f\in L^p(\sigma)$, the function f is measurable with respect to σ and the integral $\int\Omega_{\alpha,\beta}^{m,n}(x-u,y-t)f(u,t)d\sigma(u,t)$ exists. Note also that the best constants $N_{p,q}^{(\alpha,\beta),(m,n),}(v,w)$ and $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n),}(\sigma,\omega)$ coincide under the standard identifications in (3.2), and this accounts for the use of blackboard bold font to differentiate the two best constants.

A necessary condition for (3.3) to hold is the finiteness of the corresponding product fractional characteristic $\mathbb{A}_{p,q}^{(\alpha,m),(\beta,n)}(\sigma,\omega)$ of the weights,

$$\begin{aligned}\mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega) &\equiv \sup_{I\subset\mathbb{R}^m,\ J\subset\mathbb{R}^n}\left|I\right|^{\frac{\alpha}{m}+\frac{1}{q}-\frac{1}{p}}\left|J\right|^{\frac{\beta}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{\left|I\times J\right|_\omega}{\left|I\times J\right|}\right)^{\frac{1}{q}}\left(\frac{\left|I\times J\right|_\sigma}{\left|I\times J\right|}\right)^{\frac{1}{p'}}\\ &= \sup_{I\subset\mathbb{R}^m,\ J\subset\mathbb{R}^n}\left|I\right|^{\frac{\alpha}{m}-1}\left|J\right|^{\frac{\beta}{n}-1}\left(\iint_{I\times J}d\omega\right)^{\frac{1}{q}}\left(\iint_{I\times J}d\sigma\right)^{\frac{1}{p'}},\end{aligned} \tag{3.4}$$

as well as the finiteness of the larger two-tailed characteristic,

$$\begin{aligned} (3.5)\qquad & \widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega) \\ \equiv & \sup_{I\times J\subset\mathbb{R}^m\times\mathbb{R}^n} |I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\left(\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\widehat{s}_{I\times J}(x,y)^q\,d\omega(x,y)\right)^{\frac{1}{q}}\left(\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\widehat{s}_{I\times J}(u,t)^{p'}\,d\sigma(u,t)\right)^{\frac{1}{p'}}, \end{aligned}$$

where $\widehat{s}_{I\times J}(x,y)$ is the 'tail' defined in (1.3) above. When (3.2) holds, we write $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)=\mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ and $\widehat{A}_{p,q}^{(\alpha,\beta),(m,n)}(v,w)=\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$, and in the one weight case $v=w$ we simply write $A_{p,q}^{\alpha,m}(w)$ and $\widehat{A}_{p,q}^{\alpha,m}(w)$.

Lemma 3. *For $1<p,q<\infty$ and $0<\alpha\leq m$, $0<\beta\leq n$, we have*

$$\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\leq\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega). \tag{3.6}$$

Proof. To see this, we begin by noting that for any rectangle $I\times J$ we have

$$\begin{aligned} |I|^{\frac{1}{m}}|x-u| &\leq \left(|I|^{\frac{1}{m}}+|x-c_I|\right)\left(|I|^{\frac{1}{m}}+|u-c_I|\right) \text{ and } |J|^{\frac{1}{n}}|y-t|\leq\left(|J|^{\frac{1}{n}}+|y-c_J|\right)\left(|J|^{\frac{1}{n}}+|t-c_J|\right), \\ \text{i.e. } |x-u| &\leq |I|^{\frac{1}{m}}\left(1+\frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)\left(1+\frac{|u-c_I|}{|I|^{\frac{1}{m}}}\right) \text{ and } |y-t|\leq|J|^{\frac{1}{n}}\left(1+\frac{|y-c_J|}{|J|^{\frac{1}{n}}}\right)\left(1+\frac{|t-c_J|}{|J|^{\frac{1}{n}}}\right), \end{aligned}$$

and hence

$$\begin{aligned} & |x-u|^{\alpha-m}\ |y-t|^{\beta-n} \\ \geq\ & |I|^{\frac{\alpha}{m}-1}\left(1+\frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)^{\alpha-m}\left(1+\frac{|u-c_I|}{|I|^{\frac{1}{m}}}\right)^{\alpha-m}|J|^{\frac{\beta}{n}-1}\left(1+\frac{|y-c_J|}{|J|^{\frac{1}{n}}}\right)^{\beta-n}\left(1+\frac{|t-c_J|}{|J|^{\frac{1}{n}}}\right)^{\beta-n} \\ =\ & |I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\widehat{s}_{I\times J}(x,y)\,\widehat{s}_{I\times J}(u,t)\ . \end{aligned}$$

Thus for $R>0$ and $f_R(u,t)\equiv\mathbf{1}_{B(0,R)\times B(0,R)}(u,t)\,\widehat{s}_Q(u,t)^{p'-1}$, we have

$$\begin{aligned} I_{\alpha,\beta}^{m,n}(f_R\sigma)(x,y) &= \int\int_{B(0,R)\times B(0,R)}|x-u|^{\alpha-n}|y-t|^{\beta-n}\widehat{s}_{I\times J}(u,t)^{p'-1}\,d\sigma(u,t) \\ &\geq \int\int_{B(0,R)\times B(0,R)}|I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\widehat{s}_{I\times J}(x,y)\,\widehat{s}_{I\times J}(u,t)\,\widehat{s}_{I\times J}(u,t)^{p'-1}\,d\sigma(u,t) \\ &= |I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\widehat{s}_{I\times J}(x,y)\int\int_{B(0,R)\times B(0,R)}\widehat{s}_{I\times J}(u,t)^{p'}\,d\sigma(u,t). \end{aligned}$$

Substituting this into the norm inequality (3.3) gives

$$\begin{aligned} & |I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\left(\int\int_{B(0,R)\times B(0,R)}\widehat{s}_{I\times J}(u,t)^{p'}\,d\sigma(u,t)\right)\left(\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\widehat{s}_{I\times J}(x,y)^q\,d\omega(x,y)\right)^{\frac{1}{q}} \\ \leq\ & \left\{\int\int_{\mathbb{R}^m\times\mathbb{R}^n}I_{\alpha,\beta}^{m,n}(f_R)\,\sigma(x,y)^q\,d\omega(x,y)\right\}^{\frac{1}{q}} \\ \leq\ & \mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\left\{\int\int_{\mathbb{R}^m\times\mathbb{R}^n}f_R(u,t)^p\,d\sigma(u,t)\right\}^{\frac{1}{p}} \\ =\ & \mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\left(\int\int_{B(0,R)\times B(0,R)}\widehat{s}_{I\times J}(u,t)^{p'}\,d\sigma(u,t)\right)^{\frac{1}{p}}, \end{aligned}$$

and upon dividing through by $\left(\int\int_{B(0,R)\times B(0,R)} \widehat{s}_{I\times J}(u,t)^{p'}\,d\sigma(u,t)\right)^{\frac{1}{p}}$, we obtain

$$\begin{aligned}|I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}&\left(\int\int_{B(0,R)\times B(0,R)} \widehat{s}_{I\times J}(u,t)^{p'}\,d\sigma(u,t)\right)^{\frac{1}{p'}}\left(\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\widehat{s}_{I\times J}(x,y)^q\,d\omega(x,y)\right)^{\frac{1}{q}}\\ &\leq \mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega),\qquad \text{for all } R>0 \text{ and all rectangles } I\times J.\end{aligned}$$

Now take the supremum over all $R>0$ and all rectangles $I\times J$ to get (3.6). □

Remark 1. *We have the 'duality' identities* $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)=\mathbb{N}_{q',p'}^{(\alpha,\beta),(m,n)}(\omega,\sigma)$ *and* $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)=\widehat{\mathbb{A}}_{q',p'}^{(\alpha,\beta),(m,n)}(\omega,\sigma)$.

Remark 2. *Since* $\widehat{s}_{I\times J}\approx 1$ *on* $I\times J$, *we have the inequality* $\mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\lesssim\widehat{\mathbb{A}}_{q',p'}^{(\alpha,\beta),(m,n)}(\omega,\sigma)$. *In particular, we see that in the case*

$$\frac{1}{p}-\frac{1}{q}>\min\left\{\frac{\alpha}{m},\frac{\beta}{n}\right\},$$

say $\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{m}=\varepsilon>0$, *we have* $\frac{\alpha}{m}-1=-\varepsilon-\frac{1}{p'}-\frac{1}{q}$, *and so*

$$|I\times J|^{-\varepsilon}\left(\frac{1}{|I\times J|}\int\int_{I\times J}d\sigma\right)^{\frac{1}{p'}}\left(\frac{1}{|I\times J|}\int\int_{I\times J}d\omega\right)^{\frac{1}{q}}\leq\mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\lesssim\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega),$$

for all rectangles $I\times J$. *Thus the finiteness of* $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ *implies that the measures* σ *and* ω *are carried by* disjoint *sets. We shall not have much more to say regarding this case.*

4. Statements of problems and theorems, and simple proofs

In the 1-parameter setting with **one** weight and **balanced** indices, the characteristic $A_{p,q}(w)$ arose in connection with the operator norm $N_{p,q}(w)$, while in the 1-parameter setting with **two** weights and **general** indices, the two-tailed two weight characteristic $\widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)$ arose in connection with the operator norm $\mathbb{N}_{p,q}^{\alpha,m}(\sigma,\omega)$.

Thus the 2-parameter questions we investigate in this paper are these.

1. Is the finiteness of the one weight characteristic $A_{p,q}(w)$ in (4.3) below sufficient for the one weight norm inequality (3.1) with $w=v$ when the indices are balanced, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{m}=\frac{\beta}{n}$, and if so what is the dependence of the operator norm $N_{p,q}(w)$ on the characteristic $A_{p,q}(w)$?
2. Under what conditions on the indices p,q,α,β,m,n is the finiteness of the two-tailed characteristic $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ sufficient to control the operator norm $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ in the norm inequality (3.3)?
3. If the operator norm $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ fails to be controlled by the characteristic $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$, what additional side conditions on the weights σ,ω are needed for finiteness of the two-tailed characteristic $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ to imply the norm inequality (3.3)?
4. If the operator norm $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ fails to be controlled by the characteristic $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$, is it controlled by the corresponding testing or $T1$ conditions?

4.1. **A one weight theorem.** The special 'one weight' case of (3.1), namely when $v=w$, is equivalent to finiteness of the product characteristic, and we can calculate the optimal power of the characteristic.

Theorem 5. *Let* $\alpha,\beta>0$, *and suppose* $1<p,q<\infty$. *Let* $w(x,y)$ *be a nonnegative weight on* $\mathbb{R}^m\times\mathbb{R}^n$. *Then the norm inequality*

$$\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}I_{\alpha,\beta}^{m,n}f(x,y)^q\ w(x,y)^q\ dxdy\right\}^{\frac{1}{q}}\leq N_{p,q}(w)\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}f(x,y)^p\ w(x,y)^p\ dxdy\right\}^{\frac{1}{p}}\tag{4.1}$$

holds for all $f \geq 0$ if and only if *both*

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{m}=\frac{\beta}{n}, \tag{4.2}$$

and

$$A_{p,q}(w) \equiv \sup_{I\subset\mathbb{R}^m,\ J\subset\mathbb{R}^n}\left(\frac{1}{|I|\,|J|}\int\int_{I\times J} w(x,y)^q\ dxdy\right)^{\frac{1}{q}}\left(\frac{1}{|I|\,|J|}\int\int_{I\times J} w(x,y)^{-p'}\ dxdy\right)^{\frac{1}{p'}}<\infty, \tag{4.3}$$

if and only if *both (4.2) and*

$$\begin{aligned}\sup_{y\in\mathbb{R}^n}\left\{\sup_{I\subset\mathbb{R}^m}\left(\frac{1}{|I|}\int_I w^y(x)^q\ dx\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I w^y(x)^{-p'}\ dx\right)^{\frac{1}{p'}}\right\} &\leq A_{p,q}(w),\\ \sup_{x\in\mathbb{R}^m}\left\{\sup_{J\subset\mathbb{R}^n}\left(\frac{1}{|J|}\int_J w_x(y)^q\ dy\right)^{\frac{1}{q}}\left(\frac{1}{|J|}\int_J w_x(y)^{-p'}\ dy\right)^{\frac{1}{p'}}\right\} &\leq A_{p,q}(w).\end{aligned} \tag{4.4}$$

Moreover

$$N_{p,q}(w)\lesssim A_{p,q}(w)^{2+2\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}}, \tag{4.5}$$

and the exponent $2+2\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}$ *is best possible.*

There is a substitute for the case $\alpha=\beta=0$ due to R. Fefferman [Fef, see page 82], namely that the strong maximal function

$$\mathcal{M}f(x,y)\equiv\sup_{I,J}\frac{1}{|I|\,|J|}\int_I\int_J|f(x,y)|\,dxdy, \tag{4.6}$$

is bounded on the weighted space $L^p(w^p)$ if and only if $A_p(w)=A_{p,p}(w)$ is finite. This is proved in [Fef, see page 82] as an application of a rectangle covering lemma, but can also be obtained using iteration, specifically from Theorem 12 below as $\mathcal{M}$ is dominated by the iterated operators in (5.1). Another substitute for the case $\alpha=\beta=0$ is the boundedness of the double Hilbert transform

$$\mathcal{H}f(x,y)\equiv\int_I\int_J\frac{1}{x-u}\frac{1}{y-t}f(u,t)\,dudt,$$

on $L^p(w^p)$ if and only if $A_p(w)$ is finite. This result, along with corresponding results for more general product Calderón-Zygmund operators, can be easily proved using Theorem 12 below, and are left for the reader.

Before proceeding to state our main two weight theorems, we point out that, without any side conditions at all on the weights, the two-tailed product characteristic $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ *never* controls the operator norm $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ for the product fractional integral, and not even for the much smaller product dyadic fractional maximal operator $\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}$ defined on a signed measure μ by

$$\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}\mu(x,y)\equiv\sup_{\substack{R=I\times J\text{ dyadic}\\(x,y)\in R}}|I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\int\int_{I\times J}d|\mu|\,.$$

(note that $\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}\mu\leq I_{\alpha,\beta}^{m,n}\mu$ when μ is positive). This failure of $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}$ to control $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}$ in general is proved by a family of counterexamples in the Appendix below, but we content ourselves here by presenting a simple precursor of this family to show that the smaller product characteristic $\mathbb{A}_{p,q}^{(\alpha,m),(\beta,n)}(\sigma,\omega)$ doesn't control the weak type operator norm of $\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}$ (c.f. Lemma 2 in the 1-parameter setting).

Example 1. *Let* $0<\alpha,\beta<1$ *and* $1<p,q<\infty$. *Given* $0<\rho<\infty$, *define a weight pair* (σ,ω_ρ) *in the plane* $\mathbb{R}^2$ *by*

$$\sigma\equiv\delta_{(0,0)}\ \text{and}\ \omega_\rho\equiv\sum_{P\in\mathcal{P}}\delta_P\ \text{where}\ \mathcal{P}=\left\{\left(2^k,2^{-\rho k}\right)\right\}_{k=1}^{\infty}\ . \tag{4.7}$$

If $R=I\times J$ is a rectangle in the plane $\mathbb{R}\times\mathbb{R}$ with sides parallel to the axes that contains $(0,0)$ and satisfies $R\cap\mathcal{P}=\left\{\left(2^k,2^{-\rho k}\right)\right\}_{k=L+1}^{L+N}$ for some $L\geq 0$ and $N\geq 1$, then it follows that

$$|I|^{\alpha-1}|J|^{\beta-1}\left(\iint_{I\times J}d\sigma\right)^{\frac{1}{p'}}\left(\iint_{I\times J}d\omega_\rho\right)^{\frac{1}{q}}\lesssim\left(2^{L+N}\right)^{\alpha-1}\left(2^{-\rho(L+1)}\right)^{\beta-1}N^{\frac{1}{q}}1^{\frac{1}{p'}}.$$

This expression is uniformly bounded, and hence $\mathbb{A}_{p,q}^{(\alpha,\beta),(1,1)}\left(\delta_{(0,0)},\omega\right)$ is bounded, provided $\rho\leq\frac{1-\alpha}{1-\beta}$. On the other hand, the function $f(x,y)\equiv 1$ satisfies $f\in L^p(\sigma)$, while the strong dyadic fractional maximal function satisfies

$$\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}f\sigma\left(2^N,2^{-\rho N}\right)\geq\left|\left(0,2^N\right)\right|^{\alpha-1}\left|\left(0,2^{-\rho N}\right)\right|^{\beta-1}\int_{[0,2^N)}\int_{\times[0,2^{-\rho N})}fd\sigma=2^{N(\alpha-1-\rho(\beta-1))}\geq 1,$$

for all $N\geq 1$ provided $\rho\geq\frac{1-\alpha}{1-\beta}$. Thus if $\rho=\frac{1-\alpha}{1-\beta}$, the weak type operator norm of $\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}f$ is infinite:

$$\begin{aligned}\left\|\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}\right\|_{L^{q,\infty}(\omega_\rho)}&=\sup_{\substack{\lambda>0\\ \|f\|_{L^p(\sigma)}=1}}\lambda\left|\left\{x\in\mathbb{R}^n:\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}f\sigma>\lambda\right\}\right|_{\omega_\rho}^{\frac{1}{q}}\\ &\geq\frac{1}{2}\left|\left\{x\in\mathbb{R}^n:\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}f\sigma>\frac{1}{2}\right\}\right|_{\omega_\rho}^{\frac{1}{q}}=\frac{1}{2}\left(\sum_{N=1}^{\infty}\left|\left(2^N,2^{-\rho N}\right)\right|_{\omega_\rho}\right)^{\frac{1}{q}}=\infty.\end{aligned}$$

Remark 3. *It is easy to verify that when $\sigma=\delta_{(0,0)}$, the operator norm $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right)$ is in fact equivalent to the* two-tailed *characteristic $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right)$ for all measures ω. Indeed, as shown in the Appendix below, $\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right)$ and $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right)$ are each equivalent to*

$$\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}|x|^{(\alpha-m)q}|y|^{(\beta-n)q}\ d\omega(x,y)\right\}^{\frac{1}{q}}.$$

We refine the above example in the Appendix to show that $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}$ doesn't control the strong type operator norm of $\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}$.

4.1.1. *Application to a two weight half-balanced norm inequality.* Here is a two weight consequence of the one weight result above when the indices satisfy the half-balanced condition.

Theorem 6. *Suppose that $1<p,q<\infty$ and $0<\frac{\alpha}{m},\frac{\beta}{n}<1$ satisfy the half-balanced condition*

$$\frac{1}{p}-\frac{1}{q}=\min\left\{\frac{\alpha}{m},\frac{\beta}{n}\right\}.\tag{4.8}$$

1. *Suppose that $v^p\in A_p$ and $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)<\infty$. Then the two weight norm inequality (3.1) holds, and moreover*

$$\begin{aligned}&\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}I_{\alpha,\beta}^{m,n}f(x,y)^q\ w(x,y)^q\ dxdy\right\}^{\frac{1}{q}}\\ &\leq C_{p,q,\|v^p\|_{A_p}}A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)\left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}f(x,y)^p\ v(x,y)^p\ dxdy\right\}^{\frac{1}{p}}.\end{aligned}\tag{4.9}$$

2. *Suppose that $w^q\in A_q$ and $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$. Then the two weight norm inequality (3.1) holds, as well as the estimate (4.9).*

This theorem shows in particular that in the half-balanced case, the (smaller) characteristic $A_{p,q}(v,w)$ controls the two weight norm inequality (3.1) under either of the side conditions (**i**) $v^p\in A_p\times A_p$ or (**ii**) $w^q\in A_q\times A_q$. This is in stark constrast to the strictly balanced case $\frac{1}{p}-\frac{1}{q}<\min\left\{\frac{\alpha}{m},\frac{\beta}{n}\right\}$ where not even the much stronger side conditions $v^{-p'},w^q\in A_1\times A_1$ are sufficient for control of the norm inequality by $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$.

4.2. **Two weight theorems - counterexamples.** In the case $\frac{1}{p}-\frac{1}{q}<\frac{\alpha}{m}=\frac{\beta}{n}$, the above theorem fails, and *even* under the additional assumptions that *both* σ and ω are absolutely continuous and satisfy the product $A_1\times A_1$ condition, and that the *two-tailed* characteristic $\widehat{A}_{p,q}(v,w)$ is finite. Recall that the strong maximal function $\mathcal{M}f$ of f is defined by

$$\mathcal{M}f(x,y)\equiv\sup_{I\times J:\ (x,y)\in I\times J}\frac{1}{|I\times J|}\iint_{I\times J}|f|\ ,$$

and that a nonnegative weight $u(x,y)$ satisfies the product $A_1\times A_1$ condition provided there is a finite constant $C=\|u\|_{A_1\times A_1}$ such that the strong maximal function $\mathcal{M}u$ of u is pointwise dominated by Cu, i.e.

$$\mathcal{M}u(x,y)\leq Cu(x,y).$$

We state and prove the following negative result only for $m=n=1$, leaving the extension to higher dimensions for the reader. Given $0<\gamma<1$ define the (product) weight

$$s^{\gamma}(x,y)\equiv\mathcal{M}\sigma(x,y)^{1-\gamma}=|x|^{\gamma-1}|y|^{\gamma-1}, \tag{4.10}$$

and the (nonproduct) weight

$$w^{\gamma}(x,y)\equiv\sum_{k=1}^{\infty}\left[\mathcal{M}\delta_{(2^k,2^{-k})}\right]^{1-\gamma}=\sum_{k=1}^{\infty}\left|x-2^k\right|^{\gamma-1}\left|y-2^{-k}\right|^{\gamma-1}. \tag{4.11}$$

Note that both $s^{\gamma}(x,y)$ and $w^{\gamma}(x,y)$ satisfy the product $A_1\times A_1$ condition by Proposition 2 of Coifman-Rochberg [CoRo], and have reverse product doubling exponent γ - a weight u has reverse product doubling exponent $\varepsilon>0$ if

$$|sI\times tJ|_u\leq Cs^{\varepsilon m}t^{\varepsilon n}|I\times J|_u$$

for all rectangles $I\times J\subset\mathbb{R}^m\times\mathbb{R}^n$ and all $0<s,t<1$.

Theorem 7. *Suppose that $1<p,q<\infty$ and $0<\alpha<1$ satisfy*

$$\frac{1}{p}-\frac{1}{q}<\alpha.$$

There exists $1-\alpha<\gamma<1$ such that if s^{γ} and w^{γ} are as in (4.10) and (4.11) above, then we have:

1. *the weights s^{γ} and w^{γ} each satisfy the product A_1 condition and have reverse product doubling exponent γ.*
2. *the weight pair (s^{γ},w^{γ}) satisfies the fractional Muckenhoupt condition*
$$\mathbb{A}_{p,q}^{(\alpha,\alpha),(1,1)}(s^{\gamma},w^{\gamma})<\infty,$$
and even the larger two-tailed version
$$\widehat{\mathbb{A}}_{p,q}^{(\alpha,\alpha),(1,1)}(s^{\gamma},w^{\gamma})<\infty.$$
3. *the operator norm $\mathbb{N}_{p,q}^{(\alpha,\alpha),(1,1)}(s^{\gamma},w^{\gamma})$ is infinite, and even the strong dyadic fractional maximal operator $\mathcal{M}_{\alpha,\alpha}^{\mathrm{dy}}$ fails to be weak type (p,q) with respect to the weight pair (s^{γ},w^{γ}).*

4.3. **Two weight theorems - power weights.** We will see below that in the special case that *both* weights are *product* weights, i.e. $w(x,y)=w_1(x)w_2(y)$ and $v(x,y)=v_1(x)v_2(y)$, then the one parameter theory carries over fairly easily to the multiparameter setting. Despite the highly negative nature of the previous theorem for product $A_1\times A_1$ weights, the 1958 result of Stein and Weiss on power weights *does* carry over to the multiparameter setting using the sandwiching technique of Lemma 1 - where here the power weights are *not* product power weights (the theory for product power weights reduces trivially to that of the 1-parameter setting).

At this point it is instructive to observe that the kernel of the 1-parameter fractional integral $I_{\alpha+\beta}^{m+n}$ is trivially dominated by the kernel of the 2-parameter fractional integral $I_{\alpha,\beta}^{m,n}$, and hence the corresponding

1-parameter conditions in Theorem 3 are necessary for boundedness of $I_{\alpha,\beta}^{m,n}$ - namely $p \leq q$ and

$$0 < \alpha + \beta < m + n \text{ and } \gamma q < m + n \text{ and } \delta p' < m + n,$$
$$\gamma + \delta \geq 0,$$
$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha + \beta - (\gamma + \delta)}{m + n},$$

where the displayed conditions are equivalent to finiteness of the 1-parameter Muckenhoupt characteristic $A_{p,q}^{\alpha+\beta,m+n}(v_\delta, w_\gamma)$. The boundedness of $I_{\alpha,\beta}^{m,n}$ is instead given by finiteness of the rectangle Muckenhoupt characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma)$. Here now is our extension of the classical Stein-Weiss theorem to the product setting.

Theorem 8. *Let $w_\gamma(x,y) = |(x,y)|^{-\gamma} = \left(|x|^2 + |y|^2\right)^{-\frac{\gamma}{2}}$ and $v_\delta(x,y) = |(x,y)|^{\delta} = \left(|x|^2 + |y|^2\right)^{\frac{\delta}{2}}$ be a pair of nonnegative power weights on $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ with $-\infty < \gamma, \delta < \infty$. Let $1 < p, q < \infty$ and $-\infty < \alpha, \beta < \infty$. Then the following three conditions are equivalent:*

(1) *the two weight norm inequality (3.1) holds, i.e.*

(4.12)
$$\left\{\int_{\mathbb{R}^m\times\mathbb{R}^n} I_{\alpha,\beta}^{m,n} f(x,y)^q \ |(x,y)|^{-\gamma q} \ dxdy\right\}^{\frac{1}{q}} \leq N_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma)\left\{\int_{\mathbb{R}^m\times\mathbb{R}^n} f(x,y)^p \ |(x,y)|^{\delta p} \ dxdy\right\}^{\frac{1}{p}},$$

for all $f \geq 0$

(2) *the indices p, q satisfy*

$$p \leq q, \tag{4.13}$$

and the Muckenhoupt characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma)$ is finite, i.e.

$$\sup_{I\subset\mathbb{R}^m,\ J\subset\mathbb{R}^n} |I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\left(\int\int_{I\times J}|(x,y)|^{-\gamma q}\ dxdy\right)^{\frac{1}{q}}\left(\int\int_{I\times J}|(u,t)|^{-\delta p'}\ dudt\right)^{\frac{1}{p'}} < \infty, \tag{4.14}$$

(3) *the indices satisfy (4.13) and*

$$\frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{m+n} = \frac{\alpha+\beta}{m+n}, \tag{4.15}$$

and

$$\gamma + \delta \geq 0, \tag{4.16}$$

and

$$\begin{aligned} \beta - \frac{n}{p} &< \delta \text{ and } \alpha - \frac{m}{p} < \delta \text{ when } \gamma \geq 0 \geq \delta, \\ \beta - \frac{n}{q'} &< \gamma \text{ and } \alpha - \frac{m}{q'} < \gamma \text{ when } \delta \geq 0 \geq \gamma. \end{aligned} \tag{4.17}$$

In the absence of (4.13), a charactertization in terms of indices of the finiteness of the Muckenhoupt characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma)$ is given by the following theorem, not used in this paper. For any real number t let $t_+ \equiv \max\{t,0\}$ and $t_- \equiv \max\{-t,0\}$ be the positive and negative parts of t. Note that $t = t_+ - t_-$. For the statement of the next theorem, we will use the notation $0_+ \leq A$ to mean the inequality $0 < A$.

Theorem 9. *Suppose that*

$$1 < p, q < \infty \text{ and } -\infty < \alpha, \beta < \infty,$$

and $w_\gamma(x,y) = |(x,y)|^{-\gamma}$ and $v_\delta(x,y) = |(x,y)|^{\delta}$ for $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ with $-\infty < \gamma, \delta < \infty$. Let

$$\Gamma \equiv \frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'},$$

and set

$$\begin{aligned}\triangle_{p,q}^{\gamma,\delta}(m) &\equiv \left(\gamma-\frac{m}{q}\right)_{+}+\left(\delta-\frac{m}{p'}\right)_{+} ,\\ \triangle_{p,q}^{\gamma,\delta}(n) &\equiv \left(\gamma-\frac{n}{q}\right)_{+}+\left(\delta-\frac{n}{p'}\right)_{+} .\end{aligned}$$

Then the Muckenhoupt characteristic is finite, i.e.

$$A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta,w_\gamma)<\infty,$$

if and only if *the power weights w_γ^q and $v_\delta^{-p'}$ are locally integrable,*

$$\gamma q<m+n \text{ and } \delta p'<m+n, \tag{4.18}$$

and the following power weight equality and constraint inequalities for α and β hold:

$$\Gamma = \frac{\alpha+\beta-\gamma-\delta}{m+n}, \tag{4.19}$$

$$\begin{aligned}\Gamma+\frac{\triangle_{p,q}^{\gamma,\delta}(n)}{m} &\leq \frac{\alpha}{m}\leq \Gamma+\frac{\gamma+\delta}{m}-\frac{\triangle_{p,q}^{\gamma,\delta}(m)}{m},\\ \Gamma+\frac{\triangle_{p,q}^{\gamma,\delta}(m)}{n} &\leq \frac{\beta}{n}\leq \Gamma+\frac{\gamma+\delta}{n}-\frac{\triangle_{p,q}^{\gamma,\delta}(n)}{n}.\end{aligned}$$

For the proof of Theorem 9 see the arxiv.

4.4. **Two weight theorems - product weights.** When both $v(u,t)=v_1(u)v_2(t)$ and $w(x,y)=w_1(x)w_2(y)$ are product weights, the one-parameter theory carries over fairly easily, and also for product measures $\sigma=\sigma_1\times\sigma_2$ and $\omega=\omega_1\times\omega_2$. Recall that

$$\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)=\sup_{I\subset\mathbb{R}^m,\ J\subset\mathbb{R}^n}|I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\left(\int_{\mathbb{R}^m}\int_{\times\mathbb{R}^n}\widehat{s}_{I\times J}^{q}d\omega\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^m}\int_{\times\mathbb{R}^n}\widehat{s}_{I\times J}^{p'}d\sigma\right)^{\frac{1}{p'}},$$

and so for product measures $\sigma=\sigma_1\times\sigma_2$ and $\omega=\omega_1\times\omega_2$, we have

$$\begin{aligned}\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega) &= \sup_{I\subset\mathbb{R}^m}|I|^{\frac{\alpha}{m}-1}\left(\int_I\widehat{s}_I^{q}d\omega_1\right)^{\frac{1}{q}}\left(\int_I\widehat{s}_I^{p'}d\sigma_1\right)^{\frac{1}{p'}}\cdot\sup_{J\subset\mathbb{R}^n}|J|^{\frac{\beta}{n}-1}\left(\int_J\widehat{s}_J^{q}d\omega_2\right)^{\frac{1}{q}}\left(\int_J\widehat{s}_J^{p'}d\sigma_2\right)^{\frac{1}{p'}}\\ &= \widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma_1,\omega_1)\cdot\widehat{\mathbb{A}}_{p,q}^{\beta,n}(\sigma_2,\omega_2).\end{aligned}$$

Theorem 10. *Suppose that $1<p,q<\infty$ and $0<\frac{\alpha}{m},\frac{\beta}{n}<1$. If both $\sigma=\sigma_1\times\sigma_2$ and $\omega=\omega_1\times\omega_2$ are product measures on $\mathbb{R}^m\times\mathbb{R}^n$, then the norm inequality (3.3) is characterized by the two-tailed Muckenhoupt condition (3.5), i.e.*

$$\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\approx\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega) .$$

As we will see later, the proof of this theorem follows immediately from the 1-parameter version, Theorem 2, together with the measure version of the iteration Theorem 13.

4.5. **Two weight theorems - $T1$ or testing conditions.** Our final theorem shows that not even the testing or $T1$ conditions are sufficient in general for the weighted norm inequality in the strictly subbalanced case $\frac{1}{p}-\frac{1}{q}<\alpha$. Recall that a weight u satisfies the product doubling condition if $|2R|_u\leq C|R|_u$ for all rectangles R, and that this condition implies the weaker product reverse doubling condition.

Theorem 11. *Suppose that $1<p<q<\infty$ and $0<\frac{\alpha}{m},\frac{\beta}{n}<1$ satisfy*

$$\frac{1}{p}-\frac{1}{q}<\min\left\{\frac{\alpha}{m},\frac{\beta}{n}\right\},$$

and suppose that the locally finite positive Borel measures σ and ω satisfy the product doubling condition and have product reverse doubling exponent $(\varepsilon,\varepsilon')$ that satisfies

$$1-\frac{\alpha}{m}<\varepsilon<\frac{1-\frac{\alpha}{m}}{\frac{1}{q}+\frac{1}{p'}} \text{ and } 1-\frac{\beta}{n}<\varepsilon'<\frac{1-\frac{\beta}{n}}{\frac{1}{q}+\frac{1}{p'}}. \tag{4.20}$$

Then if the two weight characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)$ *is finite, the following testing (or* $T1$*) conditions hold: for all rectangles* $R\subset\mathbb{R}^m\times\mathbb{R}^n$,

$$\left\{\int\int_{\mathbb{R}^m\times\mathbb{R}^n} I_{\alpha,\beta}^{m,n}(\mathbf{1}_R\sigma)(x,y)^q\ d\omega(x,y)\right\}^{\frac{1}{q}} \leq C_{p,q}\mathbb{A}_{p,q}^{(\alpha,m),(\beta,n)}(\sigma,\omega)\,|R|_\sigma^{\frac{1}{p}}, \tag{4.21}$$

$$\left\{\int\int_{\mathbb{R}^m\times\mathbb{R}^n} I_{\alpha,\beta}^{m,n}(\mathbf{1}_R\omega)(x,y)^{p'}\ d\sigma(x,y)\right\}^{\frac{1}{p'}} \leq C_{p,q}\mathbb{A}_{p,q}^{(\alpha,m),(\beta,n)}(\sigma,\omega)\,|R|_\omega^{\frac{1}{q'}}.$$

Remark 4. *The testing condition in the first line of (4.21) only requires the reverse doubling assumption on* σ*, while the testing condition in the second line only requires reverse doubling of* ω*.*

Corollary 1. *Suppose that*

$$0<\frac{1}{p}-\frac{1}{q}<\alpha.$$

Let the weights s^γ *and* w_ρ^γ *be as defined in (4.10) and (4.11), and suppose that the conclusions of Theorem 7 hold with* $1-\alpha<\gamma<\frac{1-\alpha}{\frac{1}{q}+\frac{1}{p'}}$*. Then the two-tailed characteristic* $\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(s^\gamma,w^\gamma)$ *is finite, and the testing conditions (4.21) hold with constant* $\mathbb{A}_{p,q}^{(\alpha,m),(\beta,n)}(s^\gamma,w^\gamma)$*, but the norm inequality (3.3) fails.*

Proof. The assumption $1-\alpha<\gamma<\frac{1-\alpha}{\frac{1}{q}+\frac{1}{p'}}$ implies (4.20) with $\varepsilon=\varepsilon'=\gamma$ and $m=n=1$, and so both Theorem 7 and Theorem 11 apply. □

This corollary points to the almost complete failure of the familiar 1-parameter two weight theory to extend to the 2-parameter setting. In the 1-parameter two weight setting the norm inequality for $p<q$ is characterized by the two-tailed characteristic and also by the testing conditions. In the 2-parameter setting, both characterizations fail, and even when both weights are assumed to satisfy the product A_1 condition, the strongest of the Muckenhoupt conditions that can be imposed on an individual weight.

The proofs of our positive results in the one weight case involve the standard techniques of iteration, Lebesgue's differentiation theorem and Minkowski's inequality, and the proof of the product version of the Stein-Weiss two power weight extension involves the sandwiching technique as well, while the proofs of the negative results in the two weight case require delicate constructions of counterexamples and a quasiorthogonality argument to derive the testing conditions from the characteristic and reverse doubling assumptions on the weights. We begin with the simpler one weight norm inequality and the special case of the two weight inequality when the weights are product weights. Some of this material generalizes naturally from product fractional integrals to *iterated operators* to which we now turn.

5. Iterated operators

The product fractional integral $I_{\alpha,\beta}^{m,n}f\equiv\Omega_{\alpha,\beta}^{m,n}*f$, where

$$\Omega_{\alpha,\beta}^{m,n}(x,y)=|x|^{\alpha-m}|y|^{\beta-n},\qquad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n,$$

is an example of an *iterated operator*. In order to precisely define what we mean by an iterated operator, we denote the collection of nonnegative measurable functions on $\mathbb{R}^n$ by

$$\mathcal{N}(\mathbb{R}^n)\equiv\{g:\mathbb{R}^n\to[0,\infty]:g\text{ is Lebesgue measurable}\},$$

and we refer to a mapping $T:\mathcal{N}(\mathbb{R}^n)\to\mathcal{N}(\mathbb{R}^n)$ from $\mathcal{N}(\mathbb{R}^n)$ to itself as an *operator* on $\mathcal{N}(\mathbb{R}^n)$, without any assumption of additional properties. If T_1 is an operator on $\mathcal{N}(\mathbb{R}^m)$, we define its *product extension* to an operator $T_1\otimes\delta_0$ on $\mathcal{N}(\mathbb{R}^m\times\mathbb{R}^n)$ by

$$(T_1\otimes\delta_0)f(x,y)=T_1f^y(x),\qquad f\in\mathcal{N}(\mathbb{R}^m\times\mathbb{R}^n),$$

and similarly, if T_2 is an operator on $\mathcal{N}(\mathbb{R}^n)$ we define its *product extension* to an operator $\delta_0\otimes T_2$ on $\mathcal{N}(\mathbb{R}^m\times\mathbb{R}^n)$ by

$$(\delta_0\otimes T_2)f(x,y)=T_2f_x(y),\qquad f\in\mathcal{N}(\mathbb{R}^m\times\mathbb{R}^n).$$

Definition 1. *If T_1 is an operator on $\mathcal{N}(\mathbb{R}^m)$ and T_2 is an operator on $\mathcal{N}(\mathbb{R}^n)$, then the composition operator*

$$T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0)$$

on $\mathcal{N}(\mathbb{R}^m \times \mathbb{R}^n)$ is called an iterated operator.

To see that $I^{m,n}_{\alpha,\beta}$ is an iterated operator, define $\Omega^m_\alpha(x) = |x|^{\alpha-m}$ and $\Omega^n_\beta(y) = |y|^{\beta-n}$ and set $I^k_\gamma g = \Omega^k_\gamma * g$. Then extend the operator I^m_α from $\mathbb{R}^m$ to the product space $\mathbb{R}^m \times \mathbb{R}^n$ by defining

$$\begin{aligned}(I^m_\alpha \otimes \delta_0) f(x,y) &= [\Omega^m_\alpha \otimes \delta_0] * f(x,y)\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \Omega^m_\alpha(x-u)\,\delta_0(y-v)\,f(u,v)\,dudv\\ &= \int_{\mathbb{R}^m} \Omega^m_\alpha(x-u)\,f(u,y)\,du\\ &= I^m_\alpha f^y(x),\end{aligned}$$

where $f^y(u) \equiv f(u,y)$, and similarly define

$$\left(\delta_0 \otimes I^n_\beta\right) f(x,y) = I^n_\beta f_x(y),$$

where $f_x(v) \equiv f(x,v)$. Then from $\Omega^{m,n}_{\alpha,\beta}(x,y) = \Omega^m_\alpha(x)\,\Omega^n_\beta(y)$ we have

$$\begin{aligned}I^{m,n}_{\alpha,\beta} f(x,y) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \Omega^{m,n}_{\alpha,\beta}(x-u,y-v)\,f(u,v)\,dudv\\ &= \int_{\mathbb{R}^n} \Omega^n_\beta(y-v)\left\{\int_{\mathbb{R}^m} \Omega^m_\alpha(x-u)\,f^v(u)\,du\right\}dv\\ &= \int_{\mathbb{R}^n} \Omega^n_\beta(y-v)\left\{I^m_\alpha f^v(x)\right\}dv\\ &= \int_{\mathbb{R}^n} \Omega^n_\beta(y-v)\left\{(I^m_\alpha \otimes \delta_0) f(x,\nu)\right\}dv\\ &= \int_{\mathbb{R}^n} \Omega^n_\beta(y-v)\left\{[(I^m_\alpha \otimes \delta_0) f]_x(\nu)\right\}dv\\ &= I^n_\beta\,[(I^m_\alpha \otimes \delta_0) f]_x(y)\\ &= \left(\delta_0 \otimes I^n_\beta\right)\circ(I^m_\alpha \otimes \delta_0) f(x,y).\end{aligned}$$

Thus

$$I^{m,n}_{\alpha,\beta} = \left(\delta_0 \otimes I^n_\beta\right)\circ(I^m_\alpha \otimes \delta_0),$$

and similarly we have

$$I^{m,n}_{\alpha,\beta} = (I^m_\alpha \otimes \delta_0)\circ\left(\delta_0 \otimes I^n_\beta\right),$$

which expresses $I^{m,n}_{\alpha,\beta}$ as an iterated operator, namely the compostion of two commuting operators $I^m_\alpha \otimes \delta_0$ and $\delta_0 \otimes I^n_\beta$.

More generally, we can consider the operator

$$Tf(x,y) = (K*f)(x,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} K(x-u,y-v)\,f(u,v)\,dudv\ ,$$

where K is a *product kernel* on $\mathbb{R}^m \times \mathbb{R}^n$,

$$K(x,y) = K_1(x)\,K_2(y), \qquad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n,$$

and obtain the factorizations

$$T = (K_1 \otimes \delta_0)\circ(\delta_0 \otimes K_2) = (\delta_0 \otimes K_2)\circ(K_1 \otimes \delta_0)$$

of T into iterated operators where

$$\begin{aligned}(K_1 \otimes \delta_0) * g(x,y) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} K_1(x-u)\,\delta_0(y-v)\,g(u,v)\,dudv\\ &= \int_{\mathbb{R}^m} K_1(x-u)\,g(u,y)\,du\\ &= K_1 * g^y(x),\end{aligned}$$

and

$$(\delta_0 \otimes K_2) * h(x,y) = K_2 * g_x(y).$$

As a final example, let $T_1 = M_{\mathbb{R}^m}$ be the Hardy-Littlewood maximal operator on $\mathbb{R}^m$ and let $T_2 = M_{\mathbb{R}^n}$ be the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Then the iterated operator

$$T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) = (\delta_0 \otimes M_{\mathbb{R}^n}) \circ (M_{\mathbb{R}^m} \otimes \delta_0) \tag{5.1}$$

is usually denoted $M_{\mathbb{R}^n}(M_{\mathbb{R}^m})$, and the other iterated operator by $M_{\mathbb{R}^m}(M_{\mathbb{R}^n})$. They both dominate the strong maximal operator $\mathcal{M}$ given in (4.6).

5.1. **One weight inequalities for iterated operators.** Define the iterated Lebesgue spaces $L^{p,q}_{m,n}$ on $\mathbb{R}^m \times \mathbb{R}^n$ by

$$\|F\|_{L^{p,q}_{m,n}} \equiv \left\| \|F\|_{L^p(\mathbb{R}^m)} \right\|_{L^q(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^m} F(x,y)^p \, dx \right\}^{\frac{q}{p}} dy \right\}^{\frac{1}{q}}.$$

In the proof of the next theorem we will use Minkowski's inequality for nonnegative functions,

$$\|F\|_{L^{p,q}_{m,n}} \leq \|F\|_{L^{q,p}_{n,m}}, \qquad \text{for all } F \geq 0 \text{ and } 1 \leq p \leq q \leq \infty,$$

which written out in full is

$$\left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} F(x,y)^p \, dy \right\}^{\frac{q}{p}} dx \right\}^{\frac{1}{q}} \leq \left\{ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^m} F(x,y)^q \, dx \right\}^{\frac{p}{q}} dy \right\}^{\frac{1}{p}}.$$

Theorem 12. *Let $1 \leq p \leq q \leq \infty$ and $T_1 : \mathcal{N}(\mathbb{R}^m) \to \mathcal{N}(\mathbb{R}^m)$ and $T_2 : \mathcal{N}(\mathbb{R}^n) \to \mathcal{N}(\mathbb{R}^n)$. Suppose that $v(x,y) \geq 0$ on $\mathbb{R}^m \times \mathbb{R}^n$ satisfies*

$$\|T_1 g\|_{L^q((v^y)^q)} \leq C_1 \|g\|_{L^p((v^y)^p)}, \qquad \text{for all } g \geq 0, \tag{5.2}$$

uniformly for $y \in \mathbb{R}^n$, and

$$\|T_2 h\|_{L^q((v_x)^q)} \leq C_2 \|h\|_{L^p((v_x)^p)}, \qquad \text{for all } h \geq 0, \tag{5.3}$$

uniformly for $x \in \mathbb{R}^m$. Then the iterated operator $T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0)$ satisfies

$$\|Tf\|_{L^q(v^q)} \leq C_1 C_2 \|f\|_{L^p(v^p)}, \qquad \text{for all } f \geq 0.$$

Proof. We have

$$\begin{aligned}
\|Tf\|_{L^q(v^q)} &= \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) f(x,y)^q \, v(x,y)^q \, dy dx \right\}^{\frac{1}{q}} \\
&= \left\{ \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^n} \left| T_2 \left[(T_1 \otimes \delta_0) f \right]_x (y) \right|^q v_x(y)^q \, dy \right] dx \right\}^{\frac{1}{q}} \\
&= \left\{ \int_{\mathbb{R}^m} \left\| T_2 \left[(T_1 \otimes \delta_0) f \right]_x \right\|^q_{L^q((v_x)^q)} dx \right\}^{\frac{1}{q}} \\
&\leq C_2 \left\{ \int_{\mathbb{R}^m} \left\| \left[(T_1 \otimes \delta_0) f \right]_x \right\|^q_{L^p((v_x)^p)} dx \right\}^{\frac{1}{q}} \\
&= C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f(x,y)^p \, v(x,y)^p \, dy \right\}^{\frac{q}{p}} dx \right\}^{\frac{1}{q}},
\end{aligned}$$

where we have used $h=[(T_1\otimes\delta_0)f]_x\geq 0$ in (5.3). Then by Minkowski's inequality applied to the nonnegative function $F=(T_1\otimes\delta_0)f(x,y)\ v(x,y)$, this is dominated by

$$\begin{aligned}
& C_2\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}(T_1\otimes\delta_0)f(x,y)^q v(x,y)^q\,dx\right\}^{\frac{p}{q}}dy\right\}^{\frac{1}{p}}\\
=\ & C_2\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}T_1f^y(x)^q v^y(x)^q\,dx\right\}^{\frac{p}{q}}dy\right\}^{\frac{1}{p}}\\
=\ & C_2\left\{\int_{\mathbb{R}^n}\|T_1f^y\|^p_{L^q((v^y)^q)}\,dy\right\}^{\frac{1}{p}}\\
\leq\ & C_2C_1\left\{\int_{\mathbb{R}^n}\|f^y\|^p_{L^p((v^y)^p)}\,dy\right\}^{\frac{1}{p}}=C_2C_1\|f\|_{L^p(v^p)}\ ,
\end{aligned}$$

where we have used $g=f^y\geq 0$ in (5.2). □

The following porisms, or 'corollaries of the proof', of Theorem 12 will find application in proving Theorem 6 below.

Porism1: If we replace (5.3) with the more general *two weight* inequality

$$\|T_2h\|_{L^q((w_x)^q)}\leq C_2\|h\|_{L^p((v_x)^p)}\ ,\qquad\text{for all } h\geq 0, \tag{5.4}$$

for some weight $w(x,y)\geq 0$ on $\mathbb{R}^m\times\mathbb{R}^n$, then the iterated operator $T=(\delta_0\otimes T_2)\circ(T_1\otimes\delta_0)$ satisfies the two weight inequality

$$\|Tf\|_{L^q(w^q)}\leq C_1C_2\|f\|_{L^p(v^p)}\ ,\qquad\text{for all } f\geq 0.$$

To prove this Porism, we modify the first display in the proof of Theorem 12 to this,

$$\begin{aligned}
\|Tf\|_{L^q(w^q)} &= \left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}(\delta_0\otimes T_2)\circ(T_1\otimes\delta_0)f(x,y)^q w(x,y)^q\,dydx\right\}^{\frac{1}{q}}\\
&= \left\{\int_{\mathbb{R}^m}\left[\int_{\mathbb{R}^n}|T_2[(T_1\otimes\delta_0)f]_x(y)|^q w_x(y)^q\,dy\right]dx\right\}^{\frac{1}{q}}\\
&= \left\{\int_{\mathbb{R}^m}\|T_2[(T_1\otimes\delta_0)f]_x\|^q_{L^q((w_x)^q)}\,dx\right\}^{\frac{1}{q}}\\
&\leq C_2\left\{\int_{\mathbb{R}^m}\|[(T_1\otimes\delta_0)f]_x\|^q_{L^p((v_x)^p)}\,dx\right\}^{\frac{1}{q}}\\
&= C_2\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}(T_1\otimes\delta_0)f(x,y)^p v(x,y)^p\,dy\right\}^{\frac{q}{p}}dx\right\}^{\frac{1}{q}},
\end{aligned}$$

and then the remainder of the proof of Theorem 12 applies verbatim.

There is also the following symmetrical porism whose proof is left to the reader.

Porism2: If $\widehat{T}=(T_1\otimes\delta_0)\circ(\delta_0\otimes T_2)$ is the composition of the two iterated operators in the reverse order, and if

$$\|T_1g\|_{L^q((w^y)^q)}\leq C_1\|g\|_{L^p((v^y)^p)}\ ,\qquad\text{for all } g\geq 0,$$

uniformly for $y\in\mathbb{R}^n$, and

$$\|T_2h\|_{L^q((v_x)^q)}\leq C_2\|h\|_{L^p((v_x)^p)}\ ,\qquad\text{for all } h\geq 0,$$

uniformly for $x\in\mathbb{R}^m$, then

$$\|Tf\|_{L^q(w^q)}\leq C_1C_2\|f\|_{L^p(v^p)}\ ,\qquad\text{for all } f\geq 0.$$

In the special case that

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{m}=\frac{\beta}{n}, \tag{5.5}$$

we can exploit the equivalence of the product $A_{p,q}$ condition (4.3) with the iterated $A_{p,q}$ condition (4.3) to obtain a characterization of the one weight $L^p \to L^q$ inequality for $I_{\alpha,\beta}^{m,n}$. In fact, condition (5.5) is actually necessary for the product $A_{p,q}$ condition (4.3) to be finite.

Definition 2. *We set*

$$A_{p,q}^{(\alpha,\beta),(m,n)}(w) \equiv \sup_{I\subset\mathbb{R}^m,\ J\subset\mathbb{R}^n} |I|^{\frac{\alpha}{m}+\frac{1}{q}-\frac{1}{p}} |J|^{\frac{\beta}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{|I\times J|_{w^q}}{|I\times J|}\right)^{\frac{1}{q}} \left(\frac{|I\times J|_{w^{p'}}}{|I\times J|}\right)^{\frac{1}{p'}}.$$

Claim 1. *If $N_{p,q}^{(\alpha,\beta),(m,n),}(w)<\infty$ for some weight w, then (5.5) holds.*

Proof. Following the proof of Theorem 14 in the Appendix, we apply Hölder's inequality with dual exponents $\frac{p'+1}{p'}$ and $p'+1$ to obtain

$$\begin{aligned}
1 &= \left\{\frac{1}{|I\times J|}\int\int_{I\times J} w^{\frac{p'}{p'+1}} w^{-\frac{p'}{p'+1}}\right\}^{\frac{p'+1}{p'}} \\
&\leq \left\{\left(\frac{1}{|I\times J|}\int\int_{I\times J} w\right)^{\frac{p'}{p'+1}} \left(\frac{1}{|I\times J|}\int\int_{I\times J} w^{-p'}\right)^{\frac{1}{p'+1}}\right\}^{\frac{p'+1}{p'}} \\
&= \left(\frac{1}{|I\times J|}\int\int_{I\times J} w\right)\left(\frac{1}{|I\times J|}\int\int_{I\times J} w^{-p'}\right)^{\frac{1}{p'}} \\
&\leq \left(\frac{1}{|I\times J|}\int\int_{I\times J} w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I\times J|}\int\int_{I\times J} w^{-p'}\right)^{\frac{1}{p'}}.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
&|I|^{\frac{\alpha}{m}+\frac{1}{q}-\frac{1}{p}} |J|^{\frac{\beta}{n}+\frac{1}{q}-\frac{1}{p}} \\
\leq\ & |I|^{\frac{\alpha}{m}+\frac{1}{q}-\frac{1}{p}} |J|^{\frac{\beta}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|I\times J|}\int\int_{I\times J} w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I\times J|}\int\int_{I\times J} w^{-p'}\right)^{\frac{1}{p'}} \\
=\ & |I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\left(\int\int_{I\times J} w^q\right)^{\frac{1}{q}}\left(\int\int_{I\times J} w^{-p'}\right)^{\frac{1}{p'}} \leq N_{p,q}^{\alpha,m}(w)
\end{aligned}$$

for all rectangles $I\times J$, which implies $\frac{\alpha}{m}=\frac{1}{p}-\frac{1}{q}=\frac{\beta}{n}$ as required. □

5.2. **Proof of Theorem 5.** Now we turn to the proof of Theorem 5.

Proof of Theorem 5. By Claim 1 the balanced and diagonal condition (5.5) holds. Thus we have the necessity of (4.3) follows from Lemma 3 above:

$$\left(\frac{1}{|I|\,|J|}\int\int_{I\times J} w(u,v)^{-p'}\,dudv\right)^{\frac{1}{p'}}\left(\frac{1}{|I|\,|J|}\int\int_{I\times J} w(x,y)^q\ dxdy\right)^{\frac{1}{q}} \leq N_{p,q}(w).$$

By letting the cubes J and I shrink separately to points $y\in\mathbb{R}^n$ and $x\in\mathbb{R}^m$, we obtain (4.4). Then from the one weight theorem of Muckenhoupt and Wheeden above, Theorem 1, we conclude that both (5.2) and (5.3) hold with $T_1g=\Omega_\alpha^m * g$ and $T_2h=\Omega_\beta^n * h$. Then the conclusion of Theorem 12 proves the norm inequality (4.1).

Finally, the estimate (4.5) follows upon using the estimate of Lacey, Moen, Pérez and Torres [LaMoPeTo, LaMoPeTo], which gives

$$C_1 \lesssim A_{p,q}(w^y)^{q\max\left\{1,\frac{p'}{q}\right\}\left(1-\frac{\alpha}{m}\right)} \text{ and } C_2 \lesssim A_{p,q}(w_x)^{q\max\left\{1,\frac{p'}{q}\right\}\left(1-\frac{\beta}{n}\right)},$$

uniformly in x and y, upon noting that the characteristic $A_{p,q}$ used in [LaMoPeTo, LaMoPeTo] is the q^{th} power of that defined here. Then using that

$$q\max\left\{1,\frac{p'}{q}\right\}\left(1-\frac{\alpha}{m}\right)=\max\{q,p'\}\left(\frac{1}{q}+\frac{1}{p'}\right)=1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\},$$

we obtain (4.5). Sharpness of the exponent follows upon taking product power weights and product power functions and then arguing as in the previous work of Buckley [**?**, Buc] and Lacey, Moen, Pérez and Torres [LaMoPeTo]. □

5.3. **Two weight inequalities for product weights.** If in addition we consider *product* weights, then we can prove *two weight* versions of the theorem and corollary above using essentially the same proof strategy, namely iteration of one parameter operators.

Theorem 13. *Let $1 \leq p \leq q \leq \infty$ and $T_1 : \mathcal{N}(\mathbb{R}^m) \to \mathcal{N}(\mathbb{R}^m)$ and $T_2 : \mathcal{N}(\mathbb{R}^n) \to \mathcal{N}(\mathbb{R}^n)$. Suppose that $w(x,y) = w_1(x) w_2(y)$ and $v(x,y) = v_1(x) v_2(y)$ are both product weights, and that the weight pairs (w_1, v_1) and (w_2, v_2) on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively satisfy*

$$\|T_1 g\|_{L^q\left(w_1^q\right)} \leq C_1 \|g\|_{L^p\left(v_1^p\right)} \ , \qquad \text{for all } g \geq 0, \tag{5.6}$$

and

$$\|T_2 h\|_{L^q\left(w_2^q\right)} \leq C_2 \|h\|_{L^p\left(v_2^p\right)} \ , \qquad \text{for all } h \geq 0. \tag{5.7}$$

Then the iterated operator $T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0)$ satisfies

$$\|Tf\|_{L^q(w^q)} \leq C_1 C_2 \|f\|_{L^p(v^p)} \ , \qquad \text{for all } f \geq 0.$$

Proof. We have

$$\begin{aligned}
\|Tf\|_{L^q(w^q)} &= \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) f(x,y)^q w(x,y)^q \, dy dx \right\}^{\frac{1}{q}} \\
&= \left\{ \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^n} |T_2 [(T_1 \otimes \delta_0) f]_x (y)|^q w_1(x)^q w_2(y)^q \, dy \right] dx \right\}^{\frac{1}{q}} \\
&= \left\{ \int_{\mathbb{R}^m} \|T_2 [(T_1 \otimes \delta_0) f]_x\|_{L^q((w_2)^q)}^q w_1(x)^q \, dx \right\}^{\frac{1}{q}} \\
&\leq C_2 \left\{ \int_{\mathbb{R}^m} \|[(T_1 \otimes \delta_0) f]_x\|_{L^p((v_2)^p)}^q w_1(x)^q \, dx \right\}^{\frac{1}{q}} \\
&= C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f(x,y)^p v_2(y)^p w_1(x)^p \, dy \right\}^{\frac{q}{p}} dx \right\}^{\frac{1}{q}} ,
\end{aligned}$$

where we have used $h = [(T_1 \otimes \delta_0) f]_x \geq 0$ in (5.7). Then by Minkowski's inequality applied to the nonnegative function $F = (T_1 \otimes \delta_0) f(x,y) \ v_2(y) w_1(x)$, this is dominated by

$$\begin{aligned}
& C_2 \left\{ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^m} (T_1 \otimes \delta_0) f(x,y)^q v_2(y)^q w_1(x)^q \, dx \right\}^{\frac{p}{q}} dy \right\}^{\frac{1}{p}} \\
= \ & C_2 \left\{ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^m} T_1 f^y(x)^q w_1(x)^q \, dx \right\}^{\frac{p}{q}} v_2(y)^p \, dy \right\}^{\frac{1}{p}} \\
= \ & C_2 \left\{ \int_{\mathbb{R}^n} \|T_1 f^y\|_{L^q((w_1)^q)}^p v_2(y)^p \, dy \right\}^{\frac{1}{p}} \\
\leq \ & C_2 C_1 \left\{ \int_{\mathbb{R}^n} \|f^y\|_{L^p((v_1)^p)}^p v_2(y)^p \, dy \right\}^{\frac{1}{p}} = C_2 C_1 \|f\|_{L^p(v^p)} \ ,
\end{aligned}$$

where we have used $g = f^y \geq 0$ in (5.6). □

Corollary 2. *Suppose $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n}$. Let $w(x,y) = w_1(x) w_2(y)$ and $v(x,y) = v_1(x) v_2(y)$ be a pair of nonnegative* product *weights on $\mathbb{R}^m \times \mathbb{R}^n$. Then*

$$\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} I_{\alpha,\beta}^{m,n} f(x,y)^q \ w(x,y)^q \ dx dy \right\}^{\frac{1}{q}} \leq C_{p,q}(w,v) \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y)^p \ v(x,y)^p \ dx dy \right\}^{\frac{1}{p}} \tag{5.8}$$

for all $f \geq 0$ if and only if

$$\sup_{I \subset \mathbb{R}^m,\ J \subset \mathbb{R}^n} \left(\frac{1}{|I|\,|J|} \int \int_{\mathbb{R}^m \times \mathbb{R}^n} \left(\widehat{s}_{I,J} w \right)(x,y)^q \ dxdy \right)^{\frac{1}{q}} \times \left(\frac{1}{|I|\,|J|} \int \int_{\mathbb{R}^m \times \mathbb{R}^n} \left(\widehat{s}_{I,J} v^{-1} \right)(x,y)^{p'} \ dxdy \right)^{\frac{1}{p'}} \equiv \widetilde{A}_{p,q}(w,v) < \infty, \tag{5.9}$$

where $\widehat{s}_{I,J}(x,y) = \widehat{s}_I(x)\,\widehat{s}_J(y) = \left(1 + \frac{|x-c_I|}{|I|^{\frac{1}{m}}} \right)^{\alpha - m} \left(1 + \frac{|y-c_J|}{|J|^{\frac{1}{m}}} \right)^{\beta - n}$. *Moreover,*

$$C_{p,q}(w,v) \approx \widetilde{A}_{p,q}(w,v). \tag{5.10}$$

Proof. The necessity of (5.9) is again a standard exercise in adapting the one parameter argument in Sawyer and Wheeden to the setting of product weights.

Now we turn to the sufficiency of (5.9). Since our weights w and v are product weights, the double integrals on the left hand side of (5.9) each factor as a product of integrals over $\mathbb{R}^m$ and $\mathbb{R}^n$ separately, e.g.

$$\begin{aligned} &\left(\frac{1}{|I|\,|J|} \int \int_{\mathbb{R}^m \times \mathbb{R}^n} \left(\widehat{s}_{I,J} w \right)(x,y)^q \ dxdy \right)^{\frac{1}{q}} \\ = &\left(\frac{1}{|I|} \int_{\mathbb{R}^m} \left(\widehat{s}_I w_1 \right)(x)^q \ dx \right)^{\frac{1}{q}} \left(\frac{1}{|J|} \int_{\mathbb{R}^n} \left(\widehat{s}_J w_2 \right)(y)^q \ dy \right)^{\frac{1}{q}}. \end{aligned}$$

As a consequence, the characteristic $\widehat{A}_{p,q}(w,v)$ defined in (5.9) can be rewritten as

$$\widetilde{A}_{p,q}(w,v) = \widehat{A}_{p,q}(w_1,v_1)\,\widehat{A}_{p,q}(w_2,v_2). \tag{5.11}$$

From the two weight theorem of Sawyer and Wheeden above, Theorem 2, we conclude that (5.6) holds with $T_1 g = \Omega_\alpha^m * g$ and constant $C_1 = C\widehat{A}_{p,q}(w_1,v_1)$, and that (5.7) holds with $T_2 h = \Omega_\beta^n * h$ and constant $C_2 = C\widehat{A}_{p,q}(w_2,v_2)$. This precise dependence on $\widehat{A}_{p,q}(w_2,v_2)$ is not explicitly stated in 2, but it is easily checked by tracking the constants in the proof given there. See also the detailed proof in the appendix below. Then the conclusion of the theorem proves the norm inequality (5.8), and also the equivalence (5.10), in view of (5.11). □

Remark 5. *If we restrict the function* f *in the norm inequality in the corollary above to be a product function* $f(x,y) = f_1(x)\,f_2(y)$, *then* $\left(\Omega_{\alpha,\beta}^{m,n} * f \right)(x,y)$ *is the product function* $\Omega_\alpha^m * f_1(x)\ \ \Omega_\beta^n * f_2(y)$, *and there is a particularly trivial proof of the norm bound for such* f*:*

$$\begin{aligned} &\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left(\Omega_{\alpha,\beta}^{m,n} * f \right)(x,y)^q \ w(x,y)^q \ dxdy \right\}^{\frac{1}{q}} \\ = &\left\{ \int_{\mathbb{R}^m} \Omega_\alpha^m * f_1(x)^q \ w_1(x)^q \ dx \right\}^{\frac{1}{q}} \left\{ \int_{\mathbb{R}^n} \Omega_\beta^n * f_2(y)^q \ w_2(y)^q \ dy \right\}^{\frac{1}{q}} \\ \leq &\ C_1 \left\{ \int_{\mathbb{R}^m} f_1(x)^p \ w_1(x)^p \ dx \right\}^{\frac{1}{p}} C_2 \left\{ \int_{\mathbb{R}^n} f_2(y)^p \ w_2(y)^p \ dy \right\}^{\frac{1}{p}} \\ = &\ C_1 C_2 \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y)^p \ w(x,y)^p \ dxdy \right\}^{\frac{1}{p}}. \end{aligned}$$

Remark 6. *We have the following poinwise limit for* w_1 *above:*

$$\lim_{I \to x_0} \left(\frac{1}{|I|} \int_{\mathbb{R}^m} \left(\widehat{s}_I w_1 \right)(x)^q \ dx \right)^{\frac{1}{q}} = C w_1(x_0), \qquad \text{for a.e. } x_0 \in \mathbb{R}^m. \tag{5.12}$$

Indeed, with $I_0 \equiv \left[-\frac{1}{2}, \frac{1}{2} \right]$, *the function*

$$\widehat{s}_{I_0}(x)^q = (1+|x|)^{-(m-\alpha)q} = (1+|x|)^{-m-m\frac{q}{p'}}, \qquad x \in \mathbb{R}^m,$$

is such that the family $\left\{\frac{1}{r^m}\widehat{s}^q_{I_0}\left(\frac{\cdot}{r}\right)\right\}_{r>0}$ *is an approximate identity on* $\mathbb{R}^m$*, and thus (5.12) holds at every Lebesgue point of* w_1^q*, since* w_1^q *obviously satisfies the growth condition,*

$$\int_{\mathbb{R}^m}(1+|x|)^{-m\left(1+\frac{q}{p'}\right)}w_1(x)^q\,dx=\int_{\mathbb{R}^m}(1+|x|)^{-q(m-\alpha)}w_1(x)^q\,dx\le C<\infty, \tag{5.13}$$

when v_1 *is not the trivial weight identically infinity. Similar pointwise limits hold for the remaining three functions* w_2*,* v_1 *and* v_2*.*

6. Proof of Theorem 6

Suppose that $0<\alpha<m$, $0<\beta<n$ and without loss of generality that

$$\frac{\alpha}{m}\ge\frac{\beta}{n}=\frac{1}{p}-\frac{1}{q}>0 \tag{6.1}$$

for $1<p<q<\infty$. Let $A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)<\infty$, i.e.

$$|I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}|J|^{\frac{\beta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{p'}\right)^{\frac{1}{p'}}<\infty \tag{6.2}$$

for every $I\times J\subset\mathbb{R}^m\times\mathbb{R}^n$.

We will show that the norm inequality

$$\|wI_{\alpha\beta}f\|_{\mathbf{L}^q(\mathbb{R}^m\times\mathbb{R}^n)}\lesssim A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)\,\|fv\|_{\mathbf{L}^p(\mathbb{R}^m\times\mathbb{R}^n)} \tag{6.3}$$

holds provided that $v^p\in A_p$. Recall that $v^p\in A_p$ if

$$\left(\frac{1}{|I\times J|}\iint_{I\times J}v^p\right)^{\frac{1}{p}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{p'}\right)^{\frac{1}{p'}}<\infty \tag{6.4}$$

for every $I\times J\subset\mathbb{R}^m\times\mathbb{R}^n$. This is equivalent to $v^{-p'}\in A_{p'}\times A_{p'}$, which implies both $v^p,v^{-p'}\in RH_{r_0}\times RH_{r_0}$, the product reverse Hölder class for some exponent $r_0>1$. Of course we also have $v^p,v^{-p'}\in RH_r\times RH_r$ for all $1<r\le r_0$.

We begin with finiteness of the product characteristic

$$A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)\equiv\sup_{I\times J}|I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}|J|^{\frac{\beta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{p'}\right)^{\frac{1}{p'}}.$$

For $t>1$ we define the 'bumped up' product characteristic

$$A^{(\alpha,\beta),(m,n)}_{p,q;t}(v,w)\equiv\sup_{I\times J}|I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}|J|^{\frac{\beta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^{tq}\right)^{\frac{1}{tq}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{tp'}\right)^{\frac{1}{tp'}},$$

whose 1-parameter version arises in [SaWh, see (1.10) in part A of Theorem 1].

We start with our assumption that $\frac{\alpha}{m}\ge\frac{\beta}{n}=\frac{1}{p}-\frac{1}{q}>0$, and let

$$q_1=tp\text{ and }r_0=r_1t\text{ where both }r_1,t>1,$$

and

$$\begin{aligned}\frac{\alpha_1}{m}-\left(\frac{1}{p}-\frac{1}{q_1}\right)&=r_1\left[\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)\right],\\ \frac{\beta_1}{n}&=\frac{1}{p}-\frac{1}{q_1}.\end{aligned}$$

Now define $q_0=\frac{q}{r_0}$. Since $q>p$, we can take $r_0>t>1$ with r_0 sufficiently close to 1, so that

$$q_0=\frac{q}{r_0}>tp=q_1>p\ .$$

Finally, define

$$\begin{aligned}\left\|v^{-p'}\right\|_{RH_{r_0}\times RH_{r_0}} &\equiv \sup_{I\times J}\frac{\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{r_0p'}\right)^{\frac{1}{r_0p'}}}{\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{p'}\right)^{\frac{1}{p'}}},\\ \left\|v^{p}\right\|_{RH_{r_0}\times RH_{r_0}} &\equiv \sup_{I\times J}\frac{\left(\frac{1}{|I\times J|}\iint_{I\times J}v^{r_0p}\right)^{\frac{1}{r_0p}}}{\left(\frac{1}{|I\times J|}\iint_{I\times J}v^{p}\right)^{\frac{1}{p}}},\end{aligned}$$

to be the reverse Hölder norms of $v^{-p'}$ and v^p with exponent r_0. For convenience we set

$$B\equiv\max\left\{\left\|v^{-p'}\right\|_{RH_{r_0}\times RH_{r_0}},\left\|v^{p}\right\|_{RH_{r_0}\times RH_{r_0}}\right\}.$$

Then we have

$$\begin{aligned}A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) &\geq |I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v}\right)^{p'}\right)^{\frac{1}{p'}}\\ &\geq \frac{1}{B}|I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v^{r_1}}\right)^{p'}\right)^{\frac{1}{r_1p'}}\\ &= \frac{1}{B}|I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^{r_1tq_0}\right)^{\frac{1}{r_1tq_0}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v^{r_1}}\right)^{tp'}\right)^{\frac{1}{r_1tp'}}\\ &\geq \frac{1}{B}|I|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}w^{r_1tq_1}\right)^{\frac{1}{r_1tq_1}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v^{r_1}}\right)^{tp'}\right)^{\frac{1}{r_1tp'}}\\ A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)^{r_1} &\geq \frac{1}{B^{r_1}}|I|^{\frac{\alpha_1}{m}-\left(\frac{1}{p}-\frac{1}{q_1}\right)}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(w^{r_1}\right)^{tq_1}\right)^{\frac{1}{tq_1}}\left(\frac{1}{|I\times J|}\iint_{I\times J}\left(\frac{1}{v^{r_1}}\right)^{tp'}\right)^{\frac{1}{tp'}}.\end{aligned}$$

If we take the supremum over all rectangles $I\times J$ in this last inequality, we obtain

$$\begin{aligned}A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)^{r_1} &\geq \frac{1}{B^{r_1}}A_{p,q_1;t}^{(\alpha_1,\beta_1),(m,n)}\left(v^{r_1},w^{r_1}\right);\\ A_{p,q_1;t}^{(\alpha_1,\beta_1),(m,n)}\left(v^{r_1},w^{r_1}\right) &\leq B^{r_1}A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)^{r_1}.\end{aligned}$$

Note that we have

$$\begin{aligned}\frac{\alpha_1}{m} &\geq \frac{\beta_1}{n}=\frac{1}{p}-\frac{1}{q_1}>0,\\ 0 &< \frac{\alpha_1}{m},\frac{\beta_1}{n}<1.\end{aligned}$$

Now we let J shrink to a point $y\in\mathbb{R}^n$ to obtain

$$A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)^{r_1}\geq\frac{|I|^{\frac{\alpha_1}{m}-\left(\frac{1}{p}-\frac{1}{q_1}\right)}}{B^{r_1}}\left(\frac{1}{|I|}\iint_I\left(\left[w^{r_1}\right]^y\right)^{tq_1}\right)^{\frac{1}{tq_1}}\left(\frac{1}{|I|}\iint_I\left(\frac{1}{\left[v^{r_1}\right]^y}\right)^{tp'}\right)^{\frac{1}{tp'}},$$

or

$$\sup_{y\in\mathbb{R}^n}A_{p,q_1;t}^{\alpha_1,m}\left(\left[v^{r_1}\right]^y,\left[w^{r_1}\right]^y\right)\leq B^{r_1}A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)^{r_1}. \tag{6.5}$$

Note that we have $t>1$ and

$$\begin{aligned}\frac{\alpha_1}{m} &\geq \frac{1}{p}-\frac{1}{q_1}>0,\\ 0 &< \frac{\alpha_1}{m}<1,\end{aligned}$$

and so by the converse assertion in Theorem 1 (A) of [SaWh] we obtain

$$\left\| I^m_{\alpha_1} g \right\|_{L^q(([w^{r_1}]^y)^q)} \leq C_1 \left\| g \right\|_{L^p(([v^{r_1}]^y)^p)} \ , \qquad \text{for all } g \geq 0, \tag{6.6}$$

uniformly for a.e. $y \in \mathbb{R}^n$.

Now we turn to obtaining a corresponding estimate for the other factor. We claim that if we define $\frac{\alpha_0}{m} = \frac{1}{p} - \frac{1}{q_1}$, then

$$A^{(\alpha_0,\beta_1),(m,n)}_{p,q_1} (v^{r_1}, v^{r_1}) \leq B^{2r_1} \left\| v^p \right\|^{r_1}_{A_p} .$$

Indeed, from $v^p \in A_p$ we have since $r_0 p = r_1 t p = r_1 q_1$

$$\begin{aligned}
\left\| v^p \right\|_{A_p} &\geq \left(\frac{1}{|I \times J|} \iint_{I \times J} v^p \right)^{\frac{1}{p}} \left(\frac{1}{|I \times J|} \iint_{I \times J} \left(\frac{1}{v} \right)^{p'} \right)^{\frac{1}{p'}} \\
&\geq \frac{1}{B^2} \left(\frac{1}{|I \times J|} \iint_{I \times J} v^{r_0 p} \right)^{\frac{1}{r_0 p}} \left(\frac{1}{|I \times J|} \iint_{I \times J} \left(\frac{1}{v} \right)^{r_1 p'} \right)^{\frac{1}{r_1 p'}} \\
&= \frac{1}{B^2} \left(\frac{1}{|I \times J|} \iint_{I \times J} v^{r_1 q_1} \right)^{\frac{1}{r_1 q_1}} \left(\frac{1}{|I \times J|} \iint_{I \times J} \left(\frac{1}{v} \right)^{r_1 p'} \right)^{\frac{1}{r_1 p'}} ; \\
\left\| v^p \right\|^{r_1}_{A_p} &\geq \frac{1}{B^{2r_1}} \left(\frac{1}{|I \times J|} \iint_{I \times J} (v^{r_1})^{q_1} \right)^{\frac{1}{q_1}} \left(\frac{1}{|I \times J|} \iint_{I \times J} \left(\frac{1}{v^{r_1}} \right)^{p'} \right)^{\frac{1}{p'}} ,
\end{aligned}$$

which says

$$A^{(\alpha_0,\beta_1),(m,n)}_{p,q_1} (v^{r_1}, v^{r_1}) \leq B^{2r_1} \left\| v^p \right\|^{r_1}_{A_p} \ .$$

Now let I shrink to a point $x \in \mathbb{R}^m$ to get

$$\left\| v^p \right\|^{r_1}_{A_p} \geq \frac{1}{\left\| v^{-p'} \right\|^{2r_1}_{RH_{r_0} \times RH_{r_0}}} \left(\frac{1}{|J|} \iint_J \left([v^{r_1}]_x \right)^{q_1} \right)^{\frac{1}{q_1}} \left(\frac{1}{|J|} \iint_J \left(\frac{1}{[v^{r_1}]_x} \right)^{p'} \right)^{\frac{1}{p'}}$$

which says

$$\sup_{x \in \mathbb{R}^m} A^{\beta_1,n}_{p,q_1} \left([v^{r_1}]_x , [v^{r_1}]_x \right) \leq B^{2r_1} \left\| v^p \right\|^{r_1}_{A_p} \ . \tag{6.7}$$

Then by Theorem 5 we obtain

$$\left\| I^n_{\beta_1} h \right\|_{L^q\left(\left([v^{r_1}]_x \right)^q \right)} \leq C_2 \left\| h \right\|_{L^p\left(\left([v^{r_1}]_x \right)^p \right)} \ , \qquad \text{for all } h \geq 0, \tag{6.8}$$

uniformly for a.e. $x \in \mathbb{R}^m$. Thus with $T_1 = I^m_{\alpha_1}$ and $T_2 = I^n_{\beta_1}$, the iterated inequalities (6.6) and (6.8), together with Porism2 in Subsection 5.1, show that

$$\left\| w^{r_1} I^{m,n}_{\alpha_1,\beta_1} f \right\|_{L^{q_1}(\mathbb{R}^m \times \mathbb{R}^n)} \lesssim \left\| f v^{r_1} \right\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} . \tag{6.9}$$

On the other hand, if we choose indices q_2, α_2, β_2 to satisfy

$$\frac{\alpha_2}{m} = \frac{\beta_2}{n} = \frac{1}{p} - \frac{1}{q_2}, \tag{6.10}$$

where $q_2 > q$ is implicitly defined by

$$\frac{1}{q} = \theta \frac{1}{q_1} + (1 - \theta) \frac{1}{q_2}, \qquad 0 < \theta = \frac{1}{r} < 1, \tag{6.11}$$

then the classical Hardy-Littlewood-Sobolev inequality (on product spaces) implies

$$\left\| I^{m,n}_{\alpha_2,\beta_2} f \right\|_{L^{q_2}(\mathbb{R}^m \times \mathbb{R}^n)} \lesssim \left\| f \right\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} . \tag{6.12}$$

Direct calculations show that both

$$
\begin{aligned}
&\theta\alpha_1+(1-\theta)\alpha_2=\theta m\left(\tfrac{1}{p}-\tfrac{1}{q_1}\right)+\theta r\alpha-\theta rm\left(\tfrac{1}{p}-\tfrac{1}{q}\right)+(1-\theta)m\left(\tfrac{1}{p}-\tfrac{1}{q_2}\right)\\
&=\theta m\left(\tfrac{1}{p}-\tfrac{1}{q_1}\right)+\alpha-m\left(\tfrac{1}{p}-\tfrac{1}{q}\right)+(1-\theta)m\left(\tfrac{1}{p}-\tfrac{1}{q_2}\right)\\
&=\tfrac{m}{p}-m\left[\theta\tfrac{1}{q_1}+(1-\theta)\tfrac{1}{q_2}\right]-m\left(\tfrac{1}{p}-\tfrac{1}{q}\right)+\alpha=\alpha,
\end{aligned}
\tag{6.13}
$$

and

$$
\begin{aligned}
&\theta\beta_1+(1-\theta)\beta_2=\theta n\left(\tfrac{1}{p}-\tfrac{1}{q_1}\right)+(1-\theta)\left(\tfrac{1}{p}-\tfrac{1}{q_2}\right)\\
&=\tfrac{n}{p}-n\left[\theta\tfrac{1}{q_1}+(1-\theta)\tfrac{1}{q_2}\right]=n\left(\tfrac{1}{p}-\tfrac{1}{q}\right)=\beta.
\end{aligned}
\tag{6.14}
$$

Now let $z=\mu+i\lambda\in\mathbb{C}$. We consider the analytic operator $\mathbf{U}_{\mu+i\lambda}$ defined by

$$\mathbf{U}_{\mu+i\lambda}f\equiv f*\Omega^{\alpha,\beta}_{\mu+i\lambda}\ , \tag{6.15}$$

where

$$\Omega^{\alpha,\beta}_{\mu+i\lambda}(x,y)=\left(\frac{1}{|x|^m|y|^n}\right)^{1-\frac{(\alpha_1+\beta_1)(\mu+\mathrm{i}\lambda)+(\alpha_2+\beta_2)(1-\mu+\mathrm{i}\lambda)}{m+n}}. \tag{6.16}$$

From (6.12), we have

$$w^{r(0+i\lambda)}\mathbf{U}_{0+i\lambda}v^{-r(0+i\lambda)}\colon L^p\left(\mathbb{R}^m\times\mathbb{R}^n\right)\longrightarrow L^{q_2}\left(\mathbb{R}^m\times\mathbb{R}^n\right), \tag{6.17}$$

where the operator norm is bounded by a constant. From (6.9), we have

$$w^{r(1+i\lambda)}\mathbf{U}_{1+i\lambda}v^{-r(1+i\lambda)}\colon L^p\left(\mathbb{R}^m\times\mathbb{R}^n\right)\longrightarrow L^{q_1}\left(\mathbb{R}^m\times\mathbb{R}^n\right), \tag{6.18}$$

where the operator norm is again bounded by a constant. Moreover, the boundedness of the operators in (6.12) and (6.9) implies that the operators

$$w^{r(\mu+i\lambda)}\mathbf{U}_{\mu+i\lambda}v^{-r(\mu+i\lambda)} \tag{6.19}$$

satisfies the *admissible growth condition* in Stein [Ste2] on the strip $0\le\mu\le1$, by the maximal principle for analytic functions.

Finally, by applying Stein's interpolation theorem [Ste2, Theorem 1] and using (6.11), (6.13) and (6.14), we have

$$wI^{m,n}_{\alpha,\beta}v^{-1}\colon L^p\left(\mathbb{R}^m\times\mathbb{R}^n\right)\longrightarrow L^q\left(\mathbb{R}^m\times\mathbb{R}^n\right), \tag{6.20}$$

which proves the norm inequality (6.3).

7. Proof of Theorem 7

We begin with the definition of the product $A_1\times A_1$ condition and the product reverse doubling exponents. In the next subsection we determine conditions on the parameter γ so that the weight pair (s^γ,w^γ) defined by (4.10) and (4.11) has finite product Muckenhoupt characteristic. Then in the final subsection we determine conditions on γ for which the product fractional integral fails to be bounded, and complete the proof of Theorem 7.

7.1. **Product $A_1\times A_1$ and product reverse doubling exponents.**

It is well known that for $0<\gamma<1$, the weight $\lambda^\gamma(x)=|x|^{\gamma-1}$ on the real line $\mathbb{R}$ satisfies the A_1 condition $M\lambda^\gamma(x)\le C_\gamma\lambda^\gamma(x)$, and has reverse doubling exponent γ, i.e.

$$\frac{\int_{tI}\lambda^\gamma(x)\,dx}{\int_I\lambda^\gamma(x)\,dx}\le C_\gamma t^\gamma,\qquad\text{for all intervals } I \text{ and } 0<t<1.$$

It then follows immediately that $s^\gamma(x,y)=\lambda^\gamma(x)\lambda^\gamma(y)$ satisfies the rectangle A_1 condition $\mathcal{M}s^\gamma(x,y)\leq C_\gamma^2 s^\gamma(x,y)$ with reverse doubling exponent γ. We also have

$$\begin{aligned} M_x w^\gamma(x,y) &= M_x\left(\sum_{k=1}^\infty \lambda^\gamma\left(x-2^k\right)\lambda^\gamma\left(y-2^{-k}\right)\right)\\ &\leq \sum_{k=1}^\infty \lambda^\gamma\left(y-2^{-k}\right)M_x\left[\lambda^\gamma\left(x-2^k\right)\right]\\ &\leq \sum_{k=1}^\infty \lambda^\gamma\left(y-2^{-k}\right)C_\gamma\left[\lambda^\gamma\left(x-2^k\right)\right]=C_\gamma w^\gamma(x,y), \end{aligned}$$

and hence

$$\mathcal{M}w^\gamma(x,y)\leq M_yM_xw^\gamma(x,y)\leq C_\gamma M_y w^\gamma(x,y)\leq C_\gamma^2 w^\gamma(x,y),$$

which shows that $w^\gamma(x,y)$ also satisfies the product $A_1\times A_1$ condition. It is an easy exercise to show that the reverse doubling exponent of w^γ is also γ.

7.2. **The Muckenhoupt characteristics.** First we estimate the weight $w^\gamma(x,y)$ for $(x,y)\in[0,\infty)\times\left[0,\frac{1}{2}\right]$. Define positive integers $M=M(x)$ and $L=L(y)$ depending on x and y respectively by

$$\left|x-2^M\right|=\inf_{N\geq 1}\left|x-2^N\right| \text{ and } \left|y-2^{-L}\right|=\inf_{N\geq 1}\left|y-2^{-\rho N}\right|.$$

Thus $M(x)\sim\log_2 x$ for $x>1$ and $L(y)\sim -\log_2 y$ for $0<y<1$. We claim that

$$w^\gamma(x,y)\approx \tag{7.1}$$
$$\begin{cases} \left(2^{(1-\rho)\frac{M(x)+L(y)}{2}}\left|\frac{x}{2^{M(x)}}-1\right|\left|\frac{y}{2^{-\rho L(y)}}-1\right|\right)^{\gamma-1} & \text{if } M(x)=L(y)\\ \left(\left|\frac{x}{2^{M(x)}}-1\right|\right)^{\gamma-1}+|M(x)-L(y)|+\left(\left|\frac{y}{2^{-\rho L(y)}}-1\right|\right)^{\gamma-1} & \text{if } M(x)<L(y)\\ 2^{(M(x)-L(y))(\gamma-1)}\left\{\left|\frac{x}{2^{M(x)}}-1\right|^{\gamma-1}+|M(x)-L(y)|+\left|\frac{y}{2^{-L(y)}}-1\right|^{\gamma-1}\right\} & \text{if } M(x)>L(y) \end{cases},$$

for $(x,y)\in[0,\infty)\times\left[0,\frac{1}{2}\right]$.

Define $[b,a]$ to be the interval $[a,b]$ if $a<b$. First we suppose that $M=M(x)=L(y)$, i.e. $xy\approx 1$. Then we have

$$\begin{aligned} w_\rho^\gamma(x,y) &= \sum_{k=1}^\infty \left|x-2^k\right|^{\gamma-1}\left|y-2^{-\rho k}\right|^{\gamma-1}\\ &= \left\{\sum_{k=1}^{M-2}+\sum_{k=M-1}^{M+1}+\sum_{k=M+2}^{\infty}\right\}\left|x-2^k\right|^{\gamma-1}\left|y-2^{-k}\right|^{\gamma-1}\equiv I+II+III. \end{aligned}$$

Now

$$\begin{aligned} I &\approx \sum_{k=1}^{M-2}\left|2^M-2^k\right|^{\gamma-1}\left|2^{-\rho M}-2^{-\rho k}\right|^{\gamma-1}\approx 2^{M(\gamma-1)}\sum_{k=1}^{M-2}2^{-k(\gamma-1)}\\ &\approx 2^{M(\gamma-1)}2^{-M(\gamma-1)}=1, \end{aligned}$$

and

$$II\approx\left|x-2^M\right|^{\gamma-1}\left|y-2^{-M}\right|^{\gamma-1}=\left|\frac{x}{2^M}-1\right|^{\gamma-1}\left|\frac{y}{2^{-\rho M}}-1\right|^{\gamma-1},$$

and

$$\begin{aligned} III &\approx \sum_{k=M+2}^\infty\left|2^M-2^k\right|^{\gamma-1}\left|2^{-M}-2^{-k}\right|^{\gamma-1}\approx\left(\sum_{k=M+2}^\infty 2^{k(\gamma-1)}\right)2^{-M(\gamma-1)}\\ &\approx 2^{M(\gamma-1)}2^{-M(\gamma-1)}=1, \end{aligned}$$

so that we have

$$w^\gamma(x,y)\approx\left|x-2^{M(x)}\right|^{\gamma-1}\left|y-2^{-L(y)}\right|^{\gamma-1},\qquad \text{if } M=M(x)=L(y).$$

Next we suppose that $M = M(x) < L(y) = L$, i.e. $xy \lesssim 1$, which is the case when the point (x, y) lies essentially *beneath* the unit hyperbola $\mathcal{H}$. Then we have

$$\begin{aligned} w^{\gamma}(x,y) &= \sum_{k=1}^{\infty} \left|x - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1} \\ \approx\ & \sum_{k=1}^{M-1} \left|2^M - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1} + \left|x - 2^M\right|^{\gamma-1} \left|2^{-L} - 2^M\right|^{\gamma-1} \\ &+ \sum_{k=M+1}^{L-1} \left|x - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1} + \left|2^M - 2^L\right|^{\gamma-1} \left|y - 2^{-L}\right|^{\gamma-1} + \sum_{k=L+1}^{\infty} \left|x - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1}, \end{aligned}$$

which we write as

$$w^{\gamma}(x,y) \approx I + II + III + IV + V.$$

We have

$$I \approx 2^{M(\gamma-1)} \sum_{k=1}^{M-1} 2^{-k(\gamma-1)} \approx 2^{M(\gamma-1)} 2^{-M(\gamma-1)} = 1,$$

and

$$II \approx \left|x - 2^M\right|^{\gamma-1} 2^{-M(\gamma-1)} = \left|\frac{x}{2^M} - 1\right|^{\gamma-1},$$

and

$$III \approx \sum_{k=M+1}^{L-1} \left|2^k\right|^{\gamma-1} \left|2^{-k}\right|^{\gamma-1} = \sum_{k=M+1}^{L-1} 1 = L - M,$$

and similarly

$$IV \approx \left|\frac{y}{2^{-L}} - 1\right|^{\gamma-1}$$

and

$$V \approx 1.$$

Thus we have

$$w_{\rho}^{\gamma}(x,y) \approx \left|\frac{x}{2^M} - 1\right|^{\gamma-1} + \left|L(y) - M(x)\right| + \left|\frac{y}{2^{-L}} - 1\right|^{\gamma-1}, \qquad xy \le \frac{1}{2}.$$

Finally we suppose that $M = M(x) > L(y) = L$, i.e. $xy \gtrsim 1$, which is the case when the point (x, y) lies *above* the unit hyperbola $\mathcal{H}$. Then the analogous calculations again yield

$$\begin{aligned} w^{\gamma}(x,y) &= \sum_{k=1}^{\infty} \left|x - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1} \\ \approx\ & \sum_{k=1}^{L-1} \left|2^M - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1} + \left|2^M - 2^L\right|^{\gamma-1} \left|y - 2^{-L}\right|^{\gamma-1} \\ &+ \sum_{k=L+1}^{M-1} \left|x - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1} + \left|x - 2^M\right|^{\gamma-1} \left|2^{-L} - 2^{-M}\right|^{\gamma-1} + \sum_{k=M+1}^{\infty} \left|x - 2^k\right|^{\gamma-1} \left|y - 2^{-k}\right|^{\gamma-1}, \end{aligned}$$

which we write as

$$w^{\gamma}(x,y) \approx I + II + III + IV + V.$$

We have

$$I \approx 2^{M(\gamma-1)} \sum_{k=1}^{L-1} 2^{-k(\gamma-1)} \approx 2^{M(\gamma-1)} 2^{-L(\gamma-1)} = 2^{(M-L)(\gamma-1)},$$

and

$$II \approx 2^{M(\gamma-1)} \left|y - 2^{-L}\right|^{\gamma-1} = 2^{(M-L)(\gamma-1)} \left|\frac{y}{2^{-L}} - 1\right|^{\gamma-1},$$

and

$$III \approx \sum_{k=L+1}^{M-1} \left|2^M\right|^{\gamma-1} \left|2^{-L}\right|^{\gamma-1} = 2^{(M-L)(\gamma-1)}(M - L),$$

and similarly

$$IV \approx \left|x-2^{M}\right|^{\gamma-1}\left|2^{-L}\right|^{\gamma-1} \qquad 2^{(M-L)(\gamma-1)}\left|\frac{x}{2^{M}}-1\right|^{\gamma-1}$$

and

$$V \approx 2^{(M-L)(\gamma-1)}.$$

Thus we have

$$w^{\gamma}(x,y) \approx 2^{(M(x)-L(y))(\gamma-1)}\left\{\left|\frac{x}{2^{M(x)}}-1\right|^{\gamma-1}+|M(x)-L(y)|+\left|\frac{y}{2^{-L(y)}}-1\right|^{\gamma-1}\right\}, \qquad xy \geq 2.$$

This completes the proof of (7.1).

7.2.1. *Finiteness of* $\widehat{\mathbb{A}}_{p,q}^{\alpha,\alpha}(s^{\gamma},w^{\gamma})$. Here we begin by defining a range for the index $0<\gamma<1$ so that the weight pair (s^{γ},w^{γ}) satisfies the $\mathbb{A}_{p,q}^{\alpha,\alpha}$ condition:

$$\mathbb{A}_{p,q}^{\alpha,\alpha}(s^{\gamma},w^{\gamma})=\sup_{I\times J}|I|^{\alpha-1}|J|^{\alpha-1}\left(\iint_{I\times J}w^{\gamma}(x,y)\,dxdy\right)^{\frac{1}{q}}\left(\iint_{I\times J}s^{\gamma}(x,y)\right)^{\frac{1}{p'}}<\infty. \tag{7.2}$$

From the assumption $\frac{1}{p}-\frac{1}{q}<\alpha$ we obtain that

$$0<\gamma_0\equiv\frac{1-\alpha}{\frac{1}{q}+\frac{1}{p'}}<1,$$

and hence that

$$\gamma_0\left(\frac{1}{q}+\frac{1}{p'}\right)=1-\alpha<\gamma_0\left(\frac{1}{q}+\frac{1}{p'}\right)+(1-\gamma_0)\min\left\{\frac{1}{q},\frac{1}{p'}\right\}.$$

By continuity we can choose $\gamma<\gamma_0$ so that

$$\gamma\left(\frac{1}{q}+\frac{1}{p'}\right)<1-\alpha<\gamma\left(\frac{1}{q}+\frac{1}{p'}\right)+(1-\gamma)\min\left\{\frac{1}{q},\frac{1}{p'}\right\},$$

which we rewrite as

$$\begin{aligned}\frac{\gamma}{q}+\frac{\gamma}{p'} &< 1-\alpha<\frac{1}{q}+\frac{\gamma}{p'},\\ \frac{\gamma}{q}+\frac{\gamma}{p'} &< 1-\alpha<\frac{\gamma}{q}+\frac{1}{p'}.\end{aligned} \tag{7.3}$$

With the restriction (7.3) on the choice of γ we will now prove (7.2).

First suppose that $R=I\times J$ contains $(0,0)$ and satisfies $R\cap\mathcal{P}=\left\{\left(2^{k},2^{-k}\right)\right\}_{k=K+1}^{K+N}$ for some $K\geq 0$. The worst case is when $I=\left[0,2^{K+N}\right]$ and $J=\left[0,2^{-(K+1)}\right]$, which we now suppose holds. We will use the following four interval estimates:

$$\begin{aligned}&\int_0^{2^T}\left|\frac{x}{2^{M(x)}}-1\right|^{\gamma-1}dx\approx\sum_{N=1}^{T}\int_{2^{N-\frac{1}{2}}}^{2^{N+\frac{1}{2}}}\left|\frac{x}{2^{N}}-1\right|^{\gamma-1}dx=\sum_{N=1}^{T}2^{-N(\gamma-1)}\int_{2^{N-\frac{1}{2}}}^{2^{N+\frac{1}{2}}}\left|x-2^{N}\right|^{\gamma-1}dx\\ \approx\ &\sum_{N=1}^{T}2^{-N(\gamma-1)}2^{N\gamma}\approx\sum_{N=1}^{T}2^{N(-(\gamma-1)+\gamma)}\approx\sum_{N=1}^{T}2^{N}\approx 2^{T},\end{aligned}$$

and

$$\int_0^{2^T}\left(2^{M(x)}\left|\frac{x}{2^{M(x)}}-1\right|\right)^{\gamma-1}dx\approx\sum_{N=1}^{T}2^{N(\gamma-1)}2^{N}\approx\sum_{N=1}^{T}2^{N\gamma}\approx 2^{T\gamma},$$

and

$$\begin{aligned}&\int_0^{2^{-T}}\left|\frac{y}{2^{-L(y)}}-1\right|^{\gamma-1}dy\approx\sum_{N=T}^{\infty}\int_{2^{-\left(N-\frac{1}{2}\right)}}^{2^{-\left(N+\frac{1}{2}\right)}}\left|\frac{y}{2^{-N}}-1\right|^{\gamma-1}dy\\ =\ &\sum_{N=T}^{\infty}2^{N(\gamma-1)}\int_{2^{-\left(N-\frac{1}{2}\right)}}^{2^{-\left(N+\frac{1}{2}\right)}}\left|y-2^{-N}\right|^{\gamma-1}dy\approx\sum_{N=T}^{\infty}2^{N(\gamma-1)}2^{-N\gamma}\approx\sum_{N=T}^{\infty}2^{-N}\approx 2^{-T},\end{aligned}$$

and

$$\int_0^{2^{-T}} 2^{-L(y)(\gamma-1)}dy \approx \sum_{N=T}^{\infty} 2^{-N(\gamma-1)}2^{-N} \approx \sum_{N=T}^{\infty} 2^{-N\gamma} \approx 2^{-\gamma T}.$$

Define $R_{M,L} \equiv I_M \times J_L = \left[2^{M-\frac{1}{2}}, 2^{M+\frac{1}{2}}\right] \times \left[2^{-\left(L+\frac{1}{2}\right)}, 2^{-\left(L-\frac{1}{2}\right)}\right]$. We will also use the following three rectangle estimates,

$$\begin{aligned}\iint_{R_{M,M}} w^{\gamma}(x,y)\,dxdy &\approx \int_{I_M}\int_{J_M}\left(\left|\frac{x}{2^{M(x)}}-1\right|\left|\frac{y}{2^{-\rho L(y)}}-1\right|\right)^{\gamma-1}dxdy \\ &\approx \left(\int_{I_M}\left|\frac{x}{2^{M}}-1\right|^{\gamma-1}dx\right)\left(\int_{J_M}\left|\frac{y}{2^{-\rho M}}-1\right|^{\gamma-1}dy\right) \\ &\approx \left(2^{-M(\gamma-1)}2^{M\gamma}\right)\left(2^{M(\gamma-1)}2^{-M\gamma}\right)=1;\end{aligned}$$

and for $M<L$,

$$\begin{aligned}\iint_{R_{M,L}} w^{\gamma}(x,y)\,dxdy &\approx \int_{I_M}\int_{J_L}\left|\frac{x}{2^{M(x)}}-1\right|^{\gamma-1}dxdy+\int_{I_M}\int_{J_L}\left|\frac{y}{2^{-\rho L(y)}}-1\right|^{\gamma-1}dxdy \\ &\approx 2^{M}2^{-L}+2^{M}2^{-L}=2^{M-L},\end{aligned}$$

and for $M>L$,

$$\begin{aligned}\iint_{R_{M,L}} w_{\rho}^{\gamma}(x,y)\,dxdy &\approx \int_{I_M}\int_{J_L}\left(2^{M}\left|\frac{x}{2^{M}}-1\right|\right)^{\gamma-1}2^{-L(\gamma-1)}dxdy \\ &\quad+\int_{I_M}\int_{J_L}2^{M(\gamma-1)}\left(2^{-L}\left|\frac{y}{2^{-L}}-1\right|\right)^{\gamma-1}dxdy \\ &\approx 2^{M\gamma}2^{-\gamma L}+2^{M\gamma}2^{-L(\gamma-1)}2^{-\gamma L}\approx 2^{(M-L)\gamma}.\end{aligned}$$

Altogether we have

$$\iint_{R_{M,L}} w_{\rho}^{\gamma}(x,y)\,dxdy \approx \begin{cases} 1 & \text{if} \quad M=L \\ 2^{M-L} & \text{if} \quad M<L \\ 2^{(M-L)\gamma} & \text{if} \quad M>L \end{cases}.$$

It now follows with $I=\left[0,2^{K+N}\right]$ and $J=\left[0,2^{-\rho(K+1)}\right]$ as above that

$$\begin{aligned}\iint_{I\times J} w^{\gamma}(x,y)\,dxdy &\approx \int\int_{[0,2^{K}]\times[0,2^{-\rho K}]} w^{\gamma}+\int\int_{[2^{K},2^{K+N}]\times[0,2^{-\rho K}]} w^{\gamma}\approx \sum_{M=1}^{K}\sum_{L=K}^{\infty}\iint_{R_{M,L}} w^{\gamma}+\sum_{M=K}^{K+N}\sum_{L=K}^{\infty}\iint_{R_{M,L}} w^{\gamma} \\ &\approx \sum_{M=1}^{K}\sum_{L=K}^{\infty}\iint_{R_{M,L}} w^{\gamma}+\sum_{M=K}^{K+N}\iint_{R_{M,M}} w^{\gamma}+\sum_{M=K}^{K+N}\sum_{L=M}^{\infty}\iint_{R_{M,L}} w^{\gamma}+\sum_{M=K}^{K+N}\sum_{L=K}^{M}\iint_{R_{M,L}} w^{\gamma} \\ &\approx \sum_{M=1}^{K}\sum_{L=K}^{\infty}2^{M-L}+\sum_{M=K}^{K+N}1+\sum_{M=K}^{K+N}\sum_{L=M+1}^{\infty}2^{M-L}+\sum_{M=K}^{K+N}\sum_{L=K}^{M-1}2^{(M-L)\gamma} \\ &\approx 1+N+N+2^{N\gamma}\approx 2^{N\gamma}.\end{aligned}$$

Thus we get

$$\begin{aligned}
&|I|^{\alpha-1}|J|^{\alpha-1}\left(\iint_{I\times J} w^{\gamma}(x,y)\,dxdy\right)^{\frac{1}{q}}\left(\iint_{I\times J} s^{\gamma}(x,y)\,dxdy\right)^{\frac{1}{p'}}\\
\lesssim\ & \left(2^{K+N}\right)^{\alpha-1}\left(2^{-(K+1)}\right)^{\alpha-1}\left(2^{N\gamma}\right)^{\frac{1}{q}}\left(\int_0^{2^{K+N}} x^{\gamma-1}dx\int_0^{2^{-\rho(K+1)}} y^{\gamma-1}dy\right)^{\frac{1}{p'}}\\
\lesssim\ & \left(2^{K+N}\right)^{\alpha-1}\left(2^{-(K+1)}\right)^{\alpha-1}\left(2^{N\gamma}\right)^{\frac{1}{q}}\left(2^{(K+N)\gamma}2^{-(K+1)\gamma}\right)^{\frac{1}{p'}}\approx 2^{N\left(\alpha-1+\frac{\gamma}{q}+\frac{\gamma}{p'}\right)},
\end{aligned}$$

which is uniformly bounded over all such rectangles $R=I\times J$ since

$$\alpha-1+\frac{\gamma}{q}+\frac{\gamma}{p'}\leq 0$$

holds by (7.3).

We must now complete the check that $\mathbb{A}_{p,q}^{\alpha,\alpha}(s^{\gamma},w^{\gamma})<\infty$ by considering the remaining rectangles R. First let us suppose that $R=[0,a]\times[0,b]$ with $0<a,b<1$. Then we have

$$\begin{aligned}
&|I|^{\alpha-1}|J|^{\alpha-1}\left(\iint_{I\times J} w^{\gamma}(x,y)\,dxdy\right)^{\frac{1}{q}}\left(\iint_{I\times J} s^{\gamma}(x,y)\right)^{\frac{1}{p'}}\\
\lesssim\ & a^{\alpha-1}b^{\alpha-1}\left(\int_0^a\left|\frac{x}{2^{M(x)}}-1\right|^{\gamma-1}dx\int_0^b dy\right)^{\frac{1}{q}}\left(a^{\gamma}b^{\gamma}\right)^{\frac{1}{p'}}\\
&+a^{\alpha-1}b^{\alpha-1}\left(\int_0^a dx\int_0^b\left(\left|\frac{y}{2^{-L(y)}}-1\right|\right)^{\gamma-1}dy\right)^{\frac{1}{q}}\left(a^{\gamma}b^{\gamma}\right)^{\frac{1}{p'}}\\
\approx\ & a^{\alpha-1}b^{\alpha-1}a^{\frac{1}{q}}b^{\frac{1}{q}}a^{\frac{\gamma}{p'}}b^{\frac{\gamma}{p'}}+a^{\alpha-1}b^{\alpha-1}a^{\frac{1}{q}}b^{\frac{1}{q}}a^{\frac{\gamma}{p'}}b^{\frac{\gamma}{p'}}\approx a^{\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}}\ b^{\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}}.
\end{aligned}$$

The final expression above is uniformly bounded for $0<a,b<1$ since

$$1-\alpha\leq\frac{1}{q}+\frac{\gamma}{p'},\tag{7.4}$$

which holds by (7.3).

Now suppose that $R=[0,a]\times[0,b]$ with $1<a<\infty$ and $0<b<\min\left\{1,\frac{1}{a}\right\}$ so that R lies beneath the hyperbola $\mathcal{H}$ and includes the origin. Then we have

$$\begin{aligned}
&|I|^{\alpha-1}|J|^{\alpha-1}\left(\iint_{I\times J} w^{\gamma}(x,y)\,dxdy\right)^{\frac{1}{q}}\left(\iint_{I\times J} s^{\gamma}(x,y)\,dxdy\right)^{\frac{1}{p'}}\\
\lesssim\ & a^{\alpha-1}b^{\alpha-1}\left(\int_0^a\left|\frac{x}{2^{M(x)}}-1\right|^{\gamma-1}dx\int_0^b dy\right)^{\frac{1}{q}}\left(a^{\gamma}b^{\gamma}\right)^{\frac{1}{p'}}\\
&+a^{\alpha-1}b^{\alpha-1}\left(\int_0^a dx\int_0^b\left|\frac{y}{2^{-\rho L(y)}}-1\right|^{\gamma-1}dy\right)^{\frac{1}{q}}\left(a^{\gamma}b^{\gamma}\right)^{\frac{1}{p'}}\\
\approx\ & a^{\alpha-1}b^{\alpha-1}a^{\frac{1}{q}}b^{\frac{1}{q}}a^{\frac{\gamma}{p'}}b^{\frac{\gamma}{p'}}=a^{\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}}\ b^{\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}},
\end{aligned}$$

which is uniformly bounded for $1<a<\infty$ and $0<b<\min\left\{1,\frac{1}{a}\right\}$ since

$$\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}\geq 0,$$

which is inequality (7.4), and since

$$a^{\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}}\left(\frac{1}{a}\right)^{\alpha-1+\frac{1}{q}+\frac{\gamma}{p'}}=1$$

is bounded for $1 < a < \infty$.

Now we check yet a final family of rectangles, namely the case $R = I \times J = \left[2^N - a, 2^N + a\right] \times \left[2^{-N} - b, 2^{-N} + b\right]$ with $0 < a, b < \infty$. When $a < 2^{N-1}$ and $b < 2^{-(N+1)}$ we have that

$$\begin{aligned} & |I|^{\alpha-1} |J|^{\alpha-1} \left(\iint_{I \times J} w^\gamma (x,y)\, dxdy \right)^{\frac{1}{q}} \left(\iint_{I \times J} s^\gamma (x,y)\, dxdy \right)^{\frac{1}{p'}} \\ \lesssim \;& a^{\alpha-1} b^{\alpha-1} \left(\int_{2^N - a}^{2^N + a} \left| \frac{x}{2^N} - 1 \right|^{\gamma-1} dx \int_{2^{-N} - b}^{2^{-N} + b} \left| \frac{y}{2^{-N}} - 1 \right|^{\gamma-1} dy \right)^{\frac{1}{q}} \left(\int_{2^N - a}^{2^N + a} |x|^{\gamma-1} dx \int_{2^{-N} - b}^{2^{-N} + b} |y|^{\gamma-1} dy \right)^{\frac{1}{p'}} \\ \approx \;& a^{\alpha-1} b^{\alpha-1} \left(2^{N(1-\gamma)} a^\gamma \right)^{\frac{1}{q}} \left(2^{-N(1-\gamma)} b^\gamma \right)^{\frac{1}{q}} \left(2^{N(\gamma-1)} a 2^{-N(\gamma-1)} b \right)^{\frac{1}{p'}} \approx a^{\alpha-1+\frac{\gamma}{q}+\frac{1}{p'}}\, b^{\alpha-1+\frac{\gamma}{q}+\frac{1}{p'}}. \end{aligned}$$

This final expression is uniformly bounded for $0 < a < 1$ and $0 < b < 2^{-\rho(N+1)}$ with N fixed, since

$$1 - \alpha \leq \frac{\gamma}{q} + \frac{1}{p'} \tag{7.5}$$

holds by (7.3). The final expression is then also uniformly bounded for $1 < a < 2^{N-1}$ and $b < 2^{-(N+1)}$ and all $N \geq 1$ since

$$2^{N\left(\alpha-1+\frac{\gamma}{q}+\frac{1}{p'}\right)}\, 2^{-N\left(\alpha-1+\frac{\gamma}{q}+\frac{1}{p'}\right)} = 1.$$

At last we consider the extreme case when $I \times J = \left[2^N - a, 2^N + a\right] \times \left[0, 2^{-(N+1)}\right]$ with $0 < a < 2^{N-1}$. Then we have

$$\begin{aligned} & |I|^{\alpha-1} |J|^{\alpha-1} \left(\iint_{I \times J} s^\gamma (x,y)\, dxdy \right)^{\frac{1}{p'}} \left(\iint_{I \times J} w^\gamma (x,y)\, dxdy \right)^{\frac{1}{q}} \\ \lesssim \;& a^{\alpha-1} 2^{-N(\alpha-1)} \left(\int_{2^N - a}^{2^N + a} x^{\gamma-1} dx \int_0^{2^{-(N+1)}} y^{\gamma-1} dy \right)^{\frac{1}{p'}} \\ & \times \left(\int_{2^N - a}^{2^N + a} \int_0^{2^{-(N+1)}} \left[\left| \frac{x}{2^N} - 1 \right|^{\gamma-1} + \left| \frac{y}{2^{-\rho L(y)}} - 1 \right|^{\gamma-1} \right] dxdy \right)^{\frac{1}{q}} \\ \approx \;& a^{\alpha-1} 2^{-N(\alpha-1)} \left(2^{N(\gamma-1)} a 2^{-N\gamma} \right)^{\frac{1}{p'}} \left(2^{N(1-\gamma)} a^\gamma 2^{-N} + a 2^{-N} \right)^{\frac{1}{q}} \\ = \;& a^{\alpha-1} 2^{-N(\alpha-1)} \left(2^{N(\gamma-1)} a 2^{-N\gamma} \right)^{\frac{1}{p'}} \left(a^\gamma 2^{-N\gamma} \right)^{\frac{1}{q}} + a^{\alpha-1} b^{\beta-1} \left(2^{N(\gamma-1)} a 2^{-N\gamma} \right)^{\frac{1}{p'}} \left(a 2^{-N} \right)^{\frac{1}{q}} \\ \approx \;& 2^{-N\frac{1}{p'}} 2^{-N\frac{\gamma}{q}} a^{\alpha-1} 2^{-N(\alpha-1)} a^{\frac{\gamma}{q}} a^{\frac{1}{p'}} + 2^{-N\frac{1}{p'}} a^{\alpha-1} 2^{-N(\alpha-1)} 2^{-N\frac{1}{q}} a^{\frac{1}{q}+\frac{1}{p'}} \\ = \;& 2^{-N\left(\frac{1}{p'}+\frac{\gamma}{q}+(\alpha-1)\right)} a^{\alpha-1+\frac{\gamma}{q}+\frac{1}{p'}} + 2^{-N\left(\frac{1}{p'}+(\alpha-1)+\frac{1}{q}\right)} a^{\alpha-1+\frac{1}{q}+\frac{1}{p'}}. \end{aligned}$$

The exponents of a above are positive since we already have $\alpha - 1 + \frac{\gamma}{q} + \frac{1}{p'} \geq 0$ from (7.3), and $\alpha - 1 + \frac{1}{q} + \frac{1}{p'}$ is even bigger. Since $a < 2^N$, we have uniform boundedness in a and N as well.

The remaining rectangles R turn out to require no further restrictions. Indeed, given any rectangle $R = I \times J$, we will now verify, albeit somewhat tediously, that there is a rectangle $R' = I' \times J'$ from one of the families of rectangles considered above, such that

$$\begin{aligned} \mathcal{A}\left(I \times J; (s^\gamma, w^\gamma)\right) \;&\equiv\; |I|^{\alpha-1} |J|^{\alpha-1} \left(\iint_{I \times J} w^\gamma (x,y)\, dxdy \right)^{\frac{1}{q}} \left(\iint_{I \times J} s^\gamma (x,y) \right)^{\frac{1}{p'}} \\ &\lesssim\; |I'|^{\alpha-1} |J'|^{\alpha-1} \left(\iint_{I' \times J'} w^\gamma (x,y)\, dxdy \right)^{\frac{1}{q}} \left(\iint_{I' \times J'} s^\gamma (x,y) \right)^{\frac{1}{p'}}. \end{aligned}$$

For this, we begin with some elementary facts on the real line. The function $\varphi_\gamma(t) \equiv \frac{1-t^\gamma}{1-t}$ is convex decreasing on $(0,\infty)$ for $0<\gamma<1$, and satisfies $\lim_{t\to 1}\varphi_\gamma(t)=\gamma$ by l'Hôspital's rule, and so in particular we have

$$\gamma < \varphi_\gamma(t) = \frac{1-t^\gamma}{1-t} < 1 \text{ for } 0<t<1.$$

The case $-A<r<R<A$: Define the function

$$\begin{aligned}
F_A^\gamma(r,R) &\equiv |(r,R)|^{\alpha-1}\left(\int_r^R |x-A|^{\gamma-1}dx\right)^{\frac{1}{q}}\left(\int_r^R |x+A|^{\gamma-1}dx\right)^{\frac{1}{p'}}\\
&= (R-r)^{\alpha-1}\frac{1}{\gamma^{\frac{1}{q}}}\left[(A-r)^\gamma-(A-R)^\gamma\right]^{\frac{1}{q}}\frac{1}{\gamma^{\frac{1}{p'}}}\left[(A+R)^\gamma-(A+r)^\gamma\right]^{\frac{1}{p'}}\\
&= \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\left[\frac{(A-r)^\gamma-(A-R)^\gamma}{(A-r)-(A-R)}\right]^{\frac{1}{q}}\left[\frac{(A+R)^\gamma-(A+r)^\gamma}{(A+R)-(A+r)}\right]^{\frac{1}{p'}}\\
&= \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\left\{(A-r)^{\frac{1}{q}}(A+R)^{\frac{1}{p'}}\right\}^{\gamma-1}\left[\frac{1-\left(\frac{A-R}{A-r}\right)^\gamma}{1-\frac{A-R}{A-r}}\right]^{\frac{1}{q}}\left[\frac{1-\left(\frac{A+r}{A+R}\right)^\gamma}{1-\frac{A+r}{A+R}}\right]^{\frac{1}{p'}}\\
&= \frac{1}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}(A+R)^{\frac{1}{p'}}\right\}^{1-\gamma}}\varphi_\gamma\left(\frac{A-R}{A-r}\right)^{\frac{1}{q}}\varphi_\gamma\left(\frac{A+r}{A+R}\right).
\end{aligned}$$

Hence we have the approximation

$$F_A^\gamma(r,R)\approx\frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}(A+R)^{\frac{1}{p'}}\right\}^{1-\gamma}}.$$

Thus for $A-r=\delta\ll 1$, we have

$$F_A^\gamma(A-\delta,R)\lesssim\frac{\delta^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{\delta^{\frac{1}{q}}A^{\frac{1}{p'}}\right\}^{1-\gamma}}=\delta^{\frac{\gamma}{q}+\frac{1}{p'}-(1-\alpha)}A^{\frac{\gamma-1}{p'}},$$

while if $A+R=\delta\ll 1$, we have

$$F_A^\gamma(r,\delta-A)\lesssim\frac{\delta^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{A^{\frac{1}{q}}\delta^{\frac{1}{p'}}\right\}^{1-\gamma}}=\delta^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}A^{\frac{\gamma-1}{q}},$$

and altogether, we see that $F_A^\gamma(r,R)$ tends to zero as either $r\nearrow A$ or $R\searrow -A$ by (7.3):

$$\frac{\gamma}{q}+\frac{\gamma}{p'}<1-\alpha<\min\left\{\frac{\gamma}{q}+\frac{1}{p'},\frac{1}{q}+\frac{\gamma}{p'}\right\}. \tag{7.6}$$

Now we compute monotonicity properties of the function

$$G_A^\gamma(r,R)\equiv\frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}(A+R)^{\frac{1}{p'}}\right\}^{1-\gamma}}.$$

We have

$$\begin{aligned}
G_A^\gamma(r,R) &= \frac{(R-r)^{\frac{1-\gamma}{q}}(R-r)^{\frac{\gamma}{q}+\frac{1}{p'}-(1-\alpha)}}{(A-r)^{\frac{1-\gamma}{q}}(A+R)^{\frac{1-\gamma}{p'}}}\\
&= \left\{\left(\frac{R-r}{A-r}\right)^{\frac{1-\gamma}{q}}(R-r)^{\frac{\gamma}{q}+\frac{1}{p'}-(1-\alpha)}\right\}\frac{1}{(A+R)^{\frac{1-\gamma}{p'}}}\\
&= \left\{\left(1-\frac{A-R}{A-r}\right)^{\frac{1-\gamma}{q}}(R-r)^{\frac{\gamma}{q}+\frac{1}{p'}-(1-\alpha)}\right\}\frac{1}{(A+R)^{\frac{1-\gamma}{p'}}},
\end{aligned}$$

which is clearly decreasing in r on the interval $(-A, R)$. Similarly,

$$\begin{aligned}
G_A^\gamma(r,R) &= \frac{(R-r)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}(R-r)^{\frac{1-\gamma}{p'}}}{(A-r)^{\frac{1-\gamma}{q}}(R+A)^{\frac{1-\gamma}{p'}}} \\
&= \frac{1}{(A-r)^{\frac{1-\gamma}{q}}}\left\{(R-r)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}\left(\frac{R-r}{R+A}\right)^{\frac{1-\gamma}{p'}}\right\} \\
&= \frac{1}{(A-r)^{\frac{1-\gamma}{q}}}\left\{(R-r)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}\left(1-\frac{r+A}{R+A}\right)^{\frac{1-\gamma}{p'}}\right\},
\end{aligned}$$

which is clearly increasing in R on the interval (r, A). Thus the function $G_A^\gamma(r,R)$ increases away from the diagonal in each coordinate direction, and it follows that the maximum is achieved on the boundary, i.e.

$$\begin{aligned}
G_A^\gamma(r,A) &= \frac{(A-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}(2A)^{\frac{1}{p'}}\right\}^{1-\gamma}} = \frac{(A-r)^{\frac{\gamma}{q}+\frac{1}{p'}-(1-\alpha)}}{(2A)^{\frac{1-\gamma}{p'}}}, \\
G_A^\gamma(-A,R) &= \frac{(R+A)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(2A)^{\frac{1}{q}}(A+R)^{\frac{1}{p'}}\right\}^{1-\gamma}} = \frac{(R+A)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}}{(2A)^{\frac{1-\gamma}{q}}}.
\end{aligned}$$

It is now clear that the maximum is achieved at $(-A, A)$, and that the maximum value of G_A^γ under the above restrictions is

$$G_A^\gamma(-A,A) = (2A)^{\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)}.$$

Note that this tends to zero as $A \nearrow \infty$.

The case $r < -A < R < A$: This time we have

$$\begin{aligned}
F_A^\gamma(r,R) &\equiv |(r,R)|^{\alpha-1}\left(\int_r^R |x-A|^{\gamma-1}\,dx\right)^{\frac{1}{q}}\left(\left\{\int_r^{-A}+\int_{-A}^R\right\}|x+A|^{\gamma-1}\,dx\right)^{\frac{1}{p'}} \\
&= (R-r)^{\alpha-1}\frac{1}{\gamma^{\frac{1}{q}}}\left[(A-r)^\gamma-(A-R)^\gamma\right]^{\frac{1}{q}}\frac{1}{\gamma^{\frac{1}{p'}}}\left[|R+A|^\gamma+|r+A|^\gamma\right]^{\frac{1}{p'}} \\
&= \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\left[\frac{(A-r)^\gamma-(A-R)^\gamma}{(A-r)-(A-R)}\right]^{\frac{1}{q}}\left[\frac{|A+R|^\gamma+|A+r|^\gamma}{|A+R|+|A+r|}\right]^{\frac{1}{p'}} \\
&= \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\left\{(A-r)^{\frac{1}{q}}|A+R|^{\frac{1}{p'}}\right\}^{\gamma-1}\left[\frac{1-\left(\frac{A-R}{A-r}\right)^\gamma}{1-\frac{A-R}{A-r}}\right]^{\frac{1}{q}}\left[\frac{1+\left|\frac{A+r}{A+R}\right|^\gamma}{1+\left|\frac{A+r}{A+R}\right|}\right]^{\frac{1}{p'}} \\
&= \frac{1}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}|A+R|^{\frac{1}{p'}}\right\}^{1-\gamma}}\varphi_\gamma\left(\frac{A-R}{A-r}\right)^{\frac{1}{q}}\left[\frac{1+\left|\frac{A+r}{A+R}\right|^\gamma}{1+\left|\frac{A+r}{A+R}\right|}\right]^{\frac{1}{p'}},
\end{aligned}$$

and so

$$\begin{aligned}
F_A^\gamma(r,R) &\approx \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}|A+R|^{\frac{1}{p'}}\right\}^{1-\gamma}}\max\left\{1,\left|\frac{A+r}{A+R}\right|^{\frac{\gamma-1}{p'}}\right\} \\
&= \frac{(|A+R|+|A+r|)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{(A-r)^{\frac{1-\gamma}{q}}}\max\left\{|A+R|^{\frac{\gamma-1}{p'}},|A+r|^{\frac{\gamma-1}{p'}}\right\} \\
&\approx \frac{(|A+R|+|A+r|)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{(A-r)^{\frac{1-\gamma}{q}}}(|A+R|+|A+r|)^{\frac{\gamma-1}{p'}} \\
&= \frac{(|A+R|+|A+r|)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}}{(2A+|A+r|)^{\frac{1-\gamma}{q}}}.
\end{aligned}$$

Thus in this case we define

$$G_A^\gamma(r,R) \equiv \frac{(|A+R|+|A+r|)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}}{(2A+|A+r|)^{\frac{1-\gamma}{q}}}.$$

Note that $G_A^\gamma(r,R)$ is increasing in R on the interval $(-A,A)$ so that

$$G_A^\gamma(r,R) \le G_A^\gamma(r,A) = \frac{(2A+|A+r|)^{\frac{1}{q}+\frac{\gamma}{p'}-(1-\alpha)}}{(2A+|A+r|)^{\frac{1-\gamma}{q}}} = (2A+|A+r|)^{\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)},$$

which is at most

$$G_A^\gamma(-A,A) = (2A)^{\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)},$$

and decays like the negative power $\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)$ of $A-r$ as $r \searrow -\infty$. Note also that for each fixed $R \in (-A,A)$, $G_A^\gamma(r,R)$ is decreasing in $|A+r|$ for $-\infty < r < -A$. Similar conclusions hold for the case $-A < r < A < R$.

The case $r < R < -A$: This time we have

$$\begin{aligned}
F_A^\gamma(r,R) &\equiv |(r,R)|^{\alpha-1}\left(\int_r^R |x-A|^{\gamma-1}dx\right)^{\frac{1}{q}}\left(\int_r^R |x+A|^{\gamma-1}dx\right)^{\frac{1}{p'}} \\
&= (R-r)^{\alpha-1}\frac{1}{\gamma^{\frac{1}{q}}}\left[(A-r)^\gamma-(A-R)^\gamma\right]^{\frac{1}{q}}\frac{1}{\gamma^{\frac{1}{p'}}}\left[|A+r|^\gamma-|A+R|^\gamma\right]^{\frac{1}{p'}} \\
&= \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\left[\frac{(A-r)^\gamma-(A-R)^\gamma}{(A-r)-(A-R)}\right]^{\frac{1}{q}}\left[\frac{|A+r|^\gamma-|A+R|^\gamma}{|A+r|-|A+R|}\right]^{\frac{1}{p'}} \\
&= \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\left\{(A-r)^{\frac{1}{q}}|A+r|^{\frac{1}{p'}}\right\}^{\gamma-1}\left[\frac{1-\left(\frac{A-R}{A-r}\right)^\gamma}{1-\frac{A-R}{A-r}}\right]^{\frac{1}{q}}\left[\frac{1-\left|\frac{A+R}{A+r}\right|^\gamma}{1-\left|\frac{A+R}{A+r}\right|}\right]^{\frac{1}{p'}} \\
&= \frac{1}{\gamma^{\frac{1}{q}+\frac{1}{p'}}}\frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}|A+r|^{\frac{1}{p'}}\right\}^{1-\gamma}}\varphi_\gamma\left(\frac{A-R}{A-r}\right)^{\frac{1}{q}}\varphi_\gamma\left(\left|\frac{A+R}{A+r}\right|\right),
\end{aligned}$$

and so

$$F_A^\gamma(r,R) \approx \frac{(R-r)^{\frac{1}{q}+\frac{1}{p'}-(1-\alpha)}}{\left\{(A-r)^{\frac{1}{q}}|A+r|^{\frac{1}{p'}}\right\}^{1-\gamma}} = (R-r)^{\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)}\left(\frac{R-r}{A-r}\right)^{\frac{1-\gamma}{q}}\left(\frac{R-r}{|A+r|}\right)^{\frac{1-\gamma}{p'}}.$$

From these calculations it is now easy to see that if $s^\gamma(x,y) = |x|^{\gamma-1}|y|^{\gamma-1}$ and $w_N^\gamma(x,y) = \left|x-2^N\right|^{\gamma-1}\left|y-2^{-N}\right|^{\gamma-1}$, then with

$$S_N \equiv \left[0,2^N\right]\times\left[0,2^{-N}\right],$$

we have for any rectangle $R = I \times J$ that

$$\begin{aligned}
\mathcal{A}(R;(s^\gamma,w_N^\gamma)) &\lesssim \mathcal{A}(S_N;(s^\gamma,w_N^\gamma)) \\
&\approx \left(2^N\right)^{\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)}\left(2^{-N}\right)^{\frac{\gamma}{q}+\frac{\gamma}{p'}-(1-\alpha)} = 1
\end{aligned}$$

is bounded in $N \ge 1$. We can also easily deduce montonicity properties for $\mathcal{A}(R;(s^\gamma,w_N^\gamma))$ that follow from the one variable monontonicity properties above.

With these estimates in hand, we can now complete our proof in four steps.

Step 1: We claim that it suffices to consider rectangles R contained in the horizontal strip $[0,\infty)\times[0,1]$.

Proof. Let P denote the bottom left vertex of the rectangle R and let Q denote the upper left vertex of the rectangle R. If P lies below the x-axis $y=0$, then we can **translate** the rectangle R vertically upward until P lies on the x-axis $y=0$ without decreasing $\mathcal{A}(R;(s^\gamma,w_N^\gamma))$ by more than a fixed positive factor. This is because all of poles of the weights s^γ and w_N^γ lie on or above the x-axis $y=0$. Now if in this translated rectangle, the upper left vertex Q lies above the horizontal line $y=1$, we can **shrink** the rectangle R vertically, keeping the bottom edge fixed on the x-axis, and only shifting the top edge downward until the top edge lies on the line $y=1$, again without decreasing $\mathcal{A}(R;(s^\gamma,w_N^\gamma))$ by more than a fixed positive factor.

This is because of the estimates proved above and since all of the poles of the weights s^γ and w_N^γ lie on or below the horizontal line $y = 1$.

Similarly, if Q lies above the horizontal line $y = 1$, then we can translate R vertically downward until Q lies on the horizontal line $y = 1$ without decreasing $\mathcal{A}(R; (s^\gamma, w_N^\gamma))$ by more than a fixed positive factor. This is because all of poles of the weights s^γ and w_N^γ lie on or below the horizontal line $y = 1$. Then if in this translated rectangle, the lower left vertex P lies below the x-axis, we can **shrink** the rectangle R vertically, keeping the upper edge fixed on the horizontal line $y = 1$, and only shifting the bottom edge upward until the bottom edge lies on the x-axis, again without decreasing $\mathcal{A}(R; (s^\gamma, w_N^\gamma))$ by more than a fixed positive factor. This is because of the estimates proved above and since all of the poles of the weights s^γ and w_N^γ lie on or above the x-axis $y = 0$.

Thus at this point we have shown that we may assume the rectangle R lies in the doubly infinite horiontal strip $(-\infty, \infty) \times [0, 1]$. If the left edge of R lies to the left of the y-axis $x = 0$, then we may shrink the rectangle horizontally, keeping the right edge fixed and only shifting the left edge until it lies on the y-axis $x = 0$, all without decreasing $\mathcal{A}(R; (s^\gamma, w_N^\gamma))$ by more than a fixed positive factor. This is because of the estimates proved above and since all of the poles of the weights s^γ and w_N^γ lie on or to the right of the y-axix $x = 0$. Thus we have repeatedly shifted and shrunk the rectangle R to a rectangle R' that is contained in the horizontal strip $[0, \infty) \times [0, 1]$ and satsifies

$$\mathcal{A}(R; (s^\gamma, w_N^\gamma)) \le C\mathcal{A}(R'; (s^\gamma, w_N^\gamma)),$$

for a constant depending only on the indices $p, q, \alpha, \beta, m, n, \gamma$. □

Step 2: We claim that it suffices to consider only rectangles R contained in the horizontal strip $[0, \infty) \times [0, 1]$ that also contain at least one of the poles from $\{(0,0)\} \cup \mathcal{P} = \left\{(0,0), (1,1), \left(2, 2^{-1}\right), ..., \left(2^N, 2^{-N}\right), ...\right\}$.

Proof. If $R \subset [0, \infty) \times [0, 1]$ doesn't contain any poles, then we can expand R by moving each edge in succession outward until either it hits a pole or hits a boundary edge of $[0, \infty) \times [0, 1]$, all without decreasing $\mathcal{A}(R; (s^\gamma, w_N^\gamma))$ by more than a fixed positive factor. This is because of the estimates proved above for intervals not containing one of the poles at $\pm A$. □

Step 3: If $R \subset [0, \infty) \times [0, 1]$ contains **exactly** one pole, then $\mathcal{A}(R; (s^\gamma, w_N^\gamma)) \le C\mathcal{A}(R'; (s^\gamma, w_N^\gamma))$ for some R' already considered in one of the families above.

Proof. This is easily verified by inspection, considering the case where the pole is $(0, 0)$ separately from the case where the pole is $\left(2^N, 2^{-N}\right)$ for some $N \ge 1$. □

Step 4: If $R \subset [0, \infty) \times [0, 1]$ contains **more** than one pole, then $\mathcal{A}(R; (s^\gamma, w_N^\gamma)) \le C\mathcal{A}(R'; (s^\gamma, w_N^\gamma))$ for some R' already considered in one of the families above.

Proof. Suppose first that R contains $(0, 0)$. If in addition R contains the consecutive points

$$\left(2^{L+1}, 2^{-(L+1)}\right), ... \left(2^{L+N}, 2^{-(L+N)}\right),$$

then we can compare R favourably with the corresponding rectangle considered above. Now suppose that R doesn't contain $(0, 0)$. Then if $R = I \times J$ contains the consecutive points $\left(2^{L+1}, 2^{-(L+1)}\right), ... \left(2^{L+N}, 2^{-(L+N)}\right)$, we can compare R favourably with a corresponding rectangle considered above. □

This completes the proof that $\mathbb{A}_{p,q}^{\alpha,\alpha}(s^\gamma, w^\gamma) < \infty$. Finally, we note that since both weights s^γ, w^γ satisfy the product $A_1 \times A_1$ condition, hence also the product reverse doubling condition, it follows from Lemma 5 in the Appendix that $\widehat{\mathbb{A}}_{p,q}^{\alpha,\alpha}(s^\gamma, w^\gamma) \approx \mathbb{A}_{p,q}^{\alpha,\alpha}(s^\gamma, w^\gamma)$.

7.3. **Failure of the strong type inequality for $\mathcal{M}_{\alpha,\alpha}^{\mathrm{dy}}$.** Assume the index γ satisfies (7.3). Now we take $0 < \tau, \eta < 1$ such that

$$\max\{\gamma, (1-\eta)p, 1-\gamma\} < \tau \le \eta < \infty, \tag{7.7}$$

and define the function

$$f_{\eta,\tau}(x, y) \equiv x^{\eta-1} y^{\tau-1} \mathbf{1}_{\mathcal{H}_*}(x, y),$$

where $\mathcal{H}_* \equiv \{(x,y) \in (1,\infty) \times (0,1) : xy \leq 1\}$. Now $f_{\eta,\tau} \in L^p(s^\gamma)$ if and only if $\iint f_{\eta,\tau}^p s^\gamma$ is finite where

$$\begin{aligned}\iint f_{\eta,\tau}^p s^\gamma &= \int_1^\infty \left(\int_0^{\frac{1}{x}} y^{(\eta-1)p} y^\tau \frac{dy}{y} \right) x^{(\eta-1)p} x^{\gamma-1} dx \\ &= \frac{1}{(\eta-1)p+\tau} \int_1^\infty \left(\frac{1}{x}\right)^{(\eta-1)p+\tau} x^{p\eta-p+\gamma-1} dx \\ &= \frac{1}{(\eta-1)p+\tau} \int_1^\infty x^{\gamma-\tau} \frac{dx}{x} \\ &\approx \frac{1}{(\eta-1)p+\tau} \frac{1}{\tau-\gamma},\end{aligned}$$

since both

$$(\eta-1)p+\tau > 0 \text{ and } \gamma-\tau<0$$

hold by (7.7).

On the other hand, the strong dyadic fractional maximal function $\mathcal{M}_{\alpha,\alpha}^{\mathrm{dy}}(f_{\eta,\tau}s^\gamma)$ satisfies for $(u,v) \in [0,2^N) \times [0,2^{-N})$,

$$\begin{aligned}\mathcal{M}_{\alpha,\alpha}^{\mathrm{dy}}(f_{\eta,\tau}s^\gamma)(u,v) &\geq \left|\left(0,2^N\right)\right|^{\alpha-1} \left|\left(0,2^{-N}\right)\right|^{\alpha-1} \int\int_{[0,2^N)\times[0,2^{-N})} f_{\eta,\tau}s^\gamma \\ &= \int_1^{2^N} \int_0^{2^{-N}} x^{\eta-1} y^{\tau-1} x^{\gamma-1} y^{\gamma-1} dxdy \\ &\approx \left(\frac{1}{\tau-1+\gamma} \frac{1}{\eta-1+\gamma} \right) 2^{N(\eta-1+\gamma)} 2^{-N(\tau-1+\gamma)} \\ &\approx 2^{N(\eta-\tau)},\end{aligned} \tag{7.8}$$

for all $N \geq 1$ since both

$$\eta-1+\gamma>0 \text{ and } \tau-1+\gamma>0$$

hold by (7.7).

Now we calculate a lower bound for the strong type operator norm of $\mathcal{M}_{\alpha,\alpha}^{\mathrm{dy}}$. We note that with

$$\begin{aligned}R_{N,N} &\equiv I_N \times J_N \equiv \left[2^{N-\frac{1}{2}}, 2^N\right] \times \left[2^{-\left(N+\frac{1}{2}\right)}, 2^{-N}\right], \\ \mathcal{H}^\natural &\equiv \left\{(x,y) :\ 1 \leq x < \infty, \frac{1}{2} \leq xy \leq 1\right\},\end{aligned}$$

we have

$$\dot{\bigcup}_{N=1}^{\infty} R_{N,N} \subset \mathcal{H}^\natural$$

and

$$\begin{aligned}|R_{N,N}|_{w^\gamma} &= \int\int_{\left\{(x,y):\ 2^{N-\frac{1}{2}} \leq x \leq 2^N, 2^{-\left(N+\frac{1}{2}\right)} \leq y \leq 2^{-N}\right\}} w^\gamma(x,y)\, dxdy \\ &\approx \int_{\{x\approx 2^N\}} \int_{\{y \approx 2^{-N}\}} \left|\frac{x}{2^N}-1\right|^{\gamma-1} \left|\frac{y}{2^{-\rho N}}-1\right|^{\gamma-1} dxdy \\ &\approx \int_{\{x\approx 2^N\}} \left|\frac{x}{2^N}-1\right|^{\gamma-1} dx \int_{\{y \approx 2^{-N}\}} \left|\frac{y}{2^{-N}}-1\right|^{\gamma-1} dy \\ &\approx 2^{-N(\gamma-1)} 2^{N\gamma} 2^{N(\gamma-1)} 2^{-N\gamma} = 1.\end{aligned}$$

We also recall the estimate (7.8),

$$\mathcal{M}_{\alpha,\alpha}^{\mathrm{dy}}(f_{\eta,\tau}s)(x,y) \approx 2^{N(\eta-\tau)}, \qquad (x,y) \in R_{N,N}\ .$$

Thus we compute

$$\int_{\mathcal{H}^\natural} \mathcal{M}^{\mathrm{dy}}_{\alpha,\alpha}\left(f_{\eta,\tau}s\right)(x,y)^q \ \ w^\gamma(x,y) \ \ dxdy \gtrsim \sum_{N=1}^{\infty} 2^{N(\eta-\tau)q} = \infty$$

since $\eta \geq \tau$ by (7.7). Note that even the *weak type* norm of $\mathcal{M}^{\mathrm{dy}}_{\alpha,\alpha}\left(f_{\eta,\tau}s\right)$ is infinite!

So altogether we have

$$f_{\eta,\tau} \in L^p\left(s^\gamma\right) \ \text{ and } \mathcal{M}^{\mathrm{dy}}_{\alpha,\alpha}\left(f_{\eta,\tau}s^\gamma\right) \notin L^q\left(w^\gamma\right) \ , \tag{7.9}$$

under the assumptions that $0 < \gamma, \eta < 1$ and

$$\eta \geq \tau > \max\left\{\gamma, (1-\eta)\,p, 1-\gamma\right\}. \tag{7.10}$$

Recall also that we have $A^{\alpha,\alpha}_{p,q}\left(s^\gamma, w^\gamma_\rho\right) < \infty$ under the assumptions that $0 < \gamma < 1$ and

$$\begin{aligned} \frac{\gamma}{q} + \frac{\gamma}{p'} &< 1-\alpha < \frac{1}{q} + \frac{\gamma}{p'}, \\ \frac{\gamma}{q} + \frac{\gamma}{p'} &< 1-\alpha < \frac{\gamma}{q} + \frac{1}{p'}. \end{aligned} \tag{7.11}$$

Thus in order to obtain a counterexample for given indices $1 < p, q < \infty$ and $0 < \alpha < 1$ that satisfy $\frac{1}{p} - \frac{1}{q} < \alpha$, we need only find indices γ, η, τ that simultaneously satisfy (7.10) and (7.11). Since $\frac{1}{p} - \frac{1}{q} < \alpha$, we can find γ satisfying (7.11) as shown above, and moreover, the argument given earlier shows that we may assume γ is arbitrarily close to but less than $\gamma_0 = \frac{1-\alpha}{\frac{1}{q}+\frac{1}{p'}}$. Since $1-\alpha < \gamma_0 < 1$ we can arrange to have $1-\alpha < \gamma < 1$ as well as (7.11). Then we choose τ so that $\max\{\gamma, 1-\gamma\} < \tau < 1$, and finally we choose $0 < \eta < 1$ so close to 1 that we have $(1-\eta)\,p < \tau \leq \eta$. This completes the proof of Theorem 7.

8. Proof of Theorem 8

Here we give the proof of our main positive two weight result, Theorem 8, beginning with the necessity of (4.13) and (4.14) for the norm inequality (4.12), i.e. the proof of (1) $\Longrightarrow$ (2). The necessity of (4.13) follows from the inequality

$$|x-u|^{\alpha-m}\,|y-t|^{\beta-n} \geq |(x-u, y-t)|^{\alpha+\beta-(m+n)}\,,$$

which shows that the 1-parameter fractional integral $I^{m+n}_{\alpha+\beta}f$ is dominated by the 2-parameter fractional integral $I^{m,n}_{\alpha,\beta}f$ when f is nonnegative. Thus Theorem 15 shows that $p \leq q$. Local integrability of the kernel is necessary for the norm inequality, and so $\alpha, \beta > 0$. The necessity of finiteness of the two-tailed Muckenhoupt characteristic $\widehat{A}^{(\alpha,\beta),(m,n)}_{p,q}\left(v_\delta, w_\gamma\right)$ now follows from Proposition 3, and then we use $A^{(\alpha,\beta),(m,n)}_{p,q}\left(v_\delta, w_\gamma\right) \leq \widehat{A}^{(\alpha,\beta),(m,n)}_{p,q}\left(v_\delta, w_\gamma\right)$.

Now we turn to proving (2) $\Longrightarrow$ (3), i.e. that (4.13) and (4.14) imply the conditions (4.15), (4.16) and (4.17) on indices, namely

$$\begin{aligned} &\frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{m+n} = \frac{\alpha+\beta}{m+n}, \\ &\gamma+\delta \geq 0, \\ &\beta - \frac{n}{p} < \delta \text{ and } \alpha - \frac{m}{p} < \delta \text{ when } \gamma \geq 0 \geq \delta, \\ &\beta - \frac{n}{q'} < \gamma \text{ and } \alpha - \frac{m}{q'} < \gamma \text{ when } \delta \geq 0 \geq \gamma. \end{aligned} \tag{8.1}$$

So assume both (4.13) and (4.14). We begin with the necessity of the equality in the top line of (8.1). This follows immediately from the calculation,

$$|tI|^{\frac{\alpha}{m}-1}\,|tJ|^{\frac{\beta}{n}-1}\left(\int\int_{tI\times tJ} |(x,y)|^{-\gamma q}\,dxdy\right)^{\frac{1}{q}}\left(\int\int_{tI\times tJ} |(x,y)|^{-\delta p'}\,dxdy\right)^{\frac{1}{p'}}$$

$$t^{\alpha-m}t^{\beta-n}t^{-\gamma+\frac{m+n}{q}}t^{-\delta+\frac{m+n}{p'}}\,|I|^{\frac{\alpha}{m}-1}\,|J|^{\frac{\beta}{n}-1}\left(\int\int_{I\times J} |(x,y)|^{-\gamma q}\,dxdy\right)^{\frac{1}{q}}\left(\int\int_{I\times J} |(x,y)|^{-\delta p'}\,dxdy\right)^{\frac{1}{p'}}$$

which implies $\alpha - m + \beta - n - \gamma + \frac{m+n}{q} - \delta + \frac{m+n}{p'} = 0$. Next we note that power weights have support equal to the entire Euclidean space, and so Remark 2 shows that

$$\frac{\alpha}{m}, \frac{\beta}{n} \geq \frac{1}{p} - \frac{1}{q},$$

and combining this with (4.15) gives the second line in (8.1). Finally, we turn to proving the necessity of the third and fourth lines in (8.1).

For this, we consider $P \times Q$ to be centered at the origin. Define the truncated cubes $P^{\varepsilon} = P \setminus \{|x| < \varepsilon\} \subset \mathbb{R}^m$ and $Q^{\varepsilon} = Q \setminus \{|y| < \varepsilon\} \subset \mathbb{R}^n$ for some $\varepsilon > 0$. Let $0 < \lambda < 1$. We set $|Q|^{\frac{1}{n}} = 1$ and $|P|^{\frac{1}{m}} = \lambda$. Suppose that $\frac{\alpha}{m} - \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Then we have

(8.2)

$$\lim_{\lambda \to 0} |P|^{\frac{\alpha}{m} - \left(\frac{1}{p} - \frac{1}{q}\right)} |Q|^{\frac{\beta}{n} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|P||Q|} \iint_{P \times Q^{\varepsilon}} \left(\frac{1}{|x| + |y|} \right)^{\gamma q} dxdy \right\}^{\frac{1}{q}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^{\varepsilon}} \left(\frac{1}{|x| + |y|} \right)^{\delta\left(\frac{p}{p-1}\right)} dxdy \right\}^{\frac{p-1}{p}}$$

$$= \lim_{\lambda \to 0} \lambda^{\alpha - m\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|P||Q|} \iint_{P \times Q^{\varepsilon}} \left(\frac{1}{|x| + |y|} \right)^{\gamma q} dxdy \right\}^{\frac{1}{q}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^{\varepsilon}} \left(\frac{1}{|x| + |y|} \right)^{\delta\left(\frac{p}{p-1}\right)} dxdy \right\}^{\frac{p-1}{p}}$$

$$= \left(\lim_{\lambda \to 0} \lambda^{\alpha - m\left(\frac{1}{p} - \frac{1}{q}\right)} \right) \left\{ \int_{Q^{\varepsilon}} \left(\frac{1}{|y|} \right)^{\gamma q} dy \right\}^{\frac{1}{q}} \left\{ \int_{Q^{\varepsilon}} \left(\frac{1}{|y|} \right)^{\delta\left(\frac{p}{p-1}\right)} dy \right\}^{\frac{p-1}{p}} = 0 \text{ for every } \varepsilon > 0.$$

Case 1: Suppose $\gamma > 0, \delta \leq 0$. Let $|Q|^{\frac{1}{n}} = 1$ and $|P|^{\frac{1}{m}} = \lambda$. Suppose $\alpha - m\left(\frac{1}{p} - \frac{1}{q}\right) = 0$. Then we have

(8.3)

$$|P|^{\frac{\alpha}{m} - \left(\frac{1}{p} - \frac{1}{q}\right)} |Q|^{\frac{\beta}{n} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|P||Q|} \iint_{P \times Q^{\varepsilon}} \left(\frac{1}{|x| + |y|} \right)^{\gamma q} dxdy \right\}^{\frac{1}{q}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^{\varepsilon}} \left(\frac{1}{|x| + |y|} \right)^{\delta\left(\frac{p}{p-1}\right)} dxdy \right\}^{\frac{p-1}{p}}$$

$$\gtrsim \left\{ \int_{Q} \left(\frac{1}{\lambda + |y|} \right)^{\gamma q} dy \right\}^{\frac{1}{q}} \left\{ \int_{Q} \left(\frac{1}{|y|} \right)^{\delta\left(\frac{p}{p-1}\right)} dy \right\}^{\frac{p-1}{p}} \gtrsim \left\{ \int_{\lambda < |y| \leq 1} \left(\frac{1}{\lambda + |y|} \right)^{\gamma q} dy \right\}^{\frac{1}{q}}.$$

A direct computation shows

$$\int_{\lambda < |y| \leq 1} \left(\frac{1}{\lambda + |y|} \right)^{\gamma q} dy \approx \ln\left(\frac{1 + \lambda}{2\lambda} \right) \text{ if } \gamma = \frac{n}{q} \tag{8.4}$$

and

$$\int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |y|} \right)^{\gamma q} dy \approx \left(\frac{1}{2\lambda} \right)^{\gamma q - n} - \left(\frac{1}{\lambda + 1} \right)^{\gamma q - n} \text{ if } \gamma > \frac{n}{q}. \tag{8.5}$$

Using (8.3), (8.4) and (8.5), and letting $\lambda \to 0$, we obtain that

$$\gamma < \frac{n}{q}. \tag{8.6}$$

On the other hand, suppose that $\alpha - m\left(\frac{1}{p}-\frac{1}{q}\right) > 0$. Then we have

(8.7)

$$
\begin{aligned}
&|P|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}|Q|^{\frac{\beta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|P||Q|}\iint_{P\times Q^{\varepsilon}}\left(\frac{1}{|x|+|y|}\right)^{\gamma q}dxdy\right\}^{\frac{1}{q}}\left\{\frac{1}{|P||Q|}\int_{P\times Q^{\varepsilon}}\left(\frac{1}{|x|+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dxdy\right\}^{\frac{p-1}{p}}\\
\gtrsim\;&(\lambda)^{\alpha-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_Q\left(\frac{1}{\lambda+|y|}\right)^{\gamma q}dy\right\}^{\frac{1}{q}}\left\{\int_Q\left(\frac{1}{|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dy\right\}^{\frac{p-1}{p}}\\
\gtrsim\;&(\lambda)^{\alpha-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{0<|y|\le\lambda}\left(\frac{1}{\lambda}\right)^{\gamma q}dy\right\}^{\frac{1}{q}}\gtrsim(\lambda)^{\frac{n}{q}-\gamma+\alpha-m\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned}
$$

Recall the estimate in (8.2). Now we note that (8.7) converges to zero as $\lambda \longrightarrow 0$. By putting (8.7) together with (8.6), we obtain

$$\gamma < \frac{n}{q} + \alpha - m\left(\frac{1}{p}-\frac{1}{q}\right). \tag{8.8}$$

The formula in (4.15) implies that (8.8) is equivalent to

$$\beta - \frac{n}{p} < \delta. \tag{8.9}$$

Switching the roles of P and Q in the argument above shows that

$$\alpha - \frac{m}{p} < \delta. \tag{8.10}$$

Case Two: Consider $\gamma \le 0, \delta > 0$. Let $|Q|^{\frac{1}{n}} = 1$ and $|P|^{\frac{1}{m}} = \lambda$. Suppose $\alpha - m\left(\frac{1}{p}-\frac{1}{q}\right) = 0$. Then we have

(8.11)

$$
\begin{aligned}
&|P|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}|Q|^{\frac{\beta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|P||Q|}\iint_{P\times Q^{\varepsilon}}\left(\frac{1}{|x|+|y|}\right)^{\gamma q}dxdy\right\}^{\frac{1}{q}}\left\{\frac{1}{|P||Q|}\int_{P\times Q^{\varepsilon}}\left(\frac{1}{|x|+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dxdy\right\}^{\frac{p-1}{p}}\\
\gtrsim\;&\left\{\int_Q\left(\frac{1}{|y|}\right)^{\gamma q}dy\right\}^{\frac{1}{q}}\left\{\int_Q\left(\frac{1}{\lambda+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dy\right\}^{\frac{p-1}{p}}\gtrsim\left\{\int_{\lambda<|y|\le 1}\left(\frac{1}{\lambda+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dy\right\}^{\frac{p-1}{p}}.
\end{aligned}
$$

A direct computation shows

$$\int_{\lambda<|y|\le 1}\left(\frac{1}{\lambda+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dy \approx \ln\left(\frac{1+\lambda}{2\lambda}\right) \text{ if } \delta = n\left(\frac{p-1}{p}\right) \tag{8.12}$$

and

$$\int_{\lambda<|y|\le 1}\left(\frac{1}{\lambda+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dy \approx \left(\frac{1}{2\lambda}\right)^{\delta\left(\frac{p}{p-1}\right)-n} - \left(\frac{1}{\lambda+1}\right)^{\delta\left(\frac{p}{p-1}\right)-n} \text{ if } \delta > n\left(\frac{p-1}{p}\right). \tag{8.13}$$

Using (8.11), (8.12) and (8.13), and letting $\lambda \longrightarrow 0$, we obtain

$$\delta < n\left(\frac{p-1}{p}\right). \tag{8.14}$$

On the other hand, suppose that $\alpha - m\left(\frac{1}{p}-\frac{1}{q}\right) > 0$. Then we have

$$(8.15)$$

$$|P|^{\frac{\alpha}{m}-\left(\frac{1}{p}-\frac{1}{q}\right)}|Q|^{\frac{\beta}{n}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|P||Q|}\iint_{P\times Q^{\varepsilon}}\left(\frac{1}{|x|+|y|}\right)^{\gamma q}dxdy\right\}^{\frac{1}{q}}\left\{\frac{1}{|P||Q|}\int_{P\times Q^{\varepsilon}}\left(\frac{1}{|x|+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dxdy\right\}^{\frac{p-1}{p}}$$

$$\gtrsim (\lambda)^{\alpha-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{Q}\left(\frac{1}{|y|}\right)^{\gamma q}dy\right\}^{\frac{1}{q}}\left\{\int_{Q}\left(\frac{1}{\lambda+|y|}\right)^{\delta\left(\frac{p}{p-1}\right)}dy\right\}^{\frac{p-1}{p}}$$

$$\gtrsim (\lambda)^{\alpha-m\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\int_{0<|y|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\delta\left(\frac{p}{p-1}\right)}dy\right\}^{\frac{p-1}{p}} \gtrsim (\lambda)^{\left(\frac{p-1}{p}\right)n-\delta+\alpha-m\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Recall the estimate in (8.2). Now we note that (8.15) converges to zero as $\lambda \longrightarrow 0$. By putting (8.15) together with (8.14), we have

$$\delta < n\left(\frac{p-1}{p}\right) + \alpha - m\left(\frac{1}{p}-\frac{1}{q}\right). \tag{8.16}$$

The formula in (4.15) implies that (8.16) is equivalent to

$$\beta - n\left(\frac{q-1}{q}\right) < \gamma. \tag{8.17}$$

Switching the roles of P and Q in the argument above shows that

$$\alpha - m\left(\frac{q-1}{q}\right) < \gamma. \tag{8.18}$$

This completes the proof that (8.1) is necessary for (4.13) and (4.14), and hence we have proved (2) $\Longrightarrow$ (3).

Now we turn to proving (3) $\Longrightarrow$ (1), i.e. that (4.13) and (8.1) are sufficient for the norm inequality (4.12). To this end we use Young's inequality

$$a^{1-\theta}b^{\theta} \leq \sqrt{a^2+b^2} \leq a+b, \tag{8.19}$$

valid for $a, b \geq 0$ and $0 \leq \theta \leq 1$, in order to define weight pairs to which the sandwiching principle can be applied. The special cases $\alpha = m$ and $\beta = n$, along with some additional exceptional cases, will be treated at the end of the proof.

8.1. **The nonexceptional cases.** We assume that $\alpha < m$ and $\beta < n$ and show here that $N_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma) < \infty$ follows from (4.13), (4.15), (4.16) and (4.17) when both $\gamma \geq 0$ and $\delta \geq 0$, and also when either $\gamma < 0$ or $\delta < 0$.

Case 1: First we suppose that $\gamma \geq 0$ and $\delta \geq 0$. Define the weight pair

$$(V(u,t), W(x,y)) \equiv \left(|u|^{\frac{\delta_1 m}{m+n}}|t|^{\frac{\delta_2 n}{m+n}}, \left(\frac{1}{|x|}\right)^{\frac{\gamma_1 m}{m+n}}\left(\frac{1}{|y|}\right)^{\frac{\gamma_2 n}{m+n}}\right)$$

where the indices δ_1, δ_2, γ_1, γ_2 satisfy

$$\frac{\delta_1 m}{m+n} + \frac{\delta_2 n}{m+n} = \delta \text{ and } \frac{\gamma_1 m}{m+n} + \frac{\gamma_2 n}{m+n} = \gamma, \tag{8.20}$$

and

$$(8.21) \qquad \begin{aligned} \frac{\alpha - \left(\frac{\gamma_1 m}{m+n} + \frac{\delta_1 m}{m+n}\right)}{m} &= \Gamma = \frac{\beta - \left(\frac{\gamma_2 n}{m+n} + \frac{\delta_2 n}{m+n}\right)}{n}, \\ \text{i.e. } \frac{\gamma_1 m}{m+n} + \frac{\delta_1 m}{m+n} &= \alpha - m\Gamma \text{ and } \frac{\gamma_2 n}{m+n} + \frac{\delta_2 n}{m+n} = \beta - n\Gamma \end{aligned}$$

Solving for

$$\triangle_1 \equiv \frac{\delta_1 m}{m+n}, \quad \triangle_2 \equiv \frac{\delta_2 n}{m+n}, \quad \Gamma_1 \equiv \frac{\gamma_1 m}{m+n}\gamma_1, \quad \Gamma_2 \equiv \frac{\gamma_2 n}{m+n},$$

we obtain the system

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \triangle_1 \\ \triangle_2 \\ \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \delta \\ \beta - n\Gamma \\ \gamma \\ \alpha - m\Gamma \end{pmatrix}, \tag{8.22}$$

which in reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \triangle_1 \\ \triangle_2 \\ \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \delta - \beta + n\Gamma \\ \beta - n\Gamma \\ \gamma \\ \alpha - m\Gamma - \delta + \beta - n\Gamma \end{pmatrix}.$$

The system is solvable since $\alpha - m\Gamma - \delta + \beta - n\Gamma = 0$ by the power weight equality in the first line of (4.19), and the general solution to the system (8.22) is thus given by

$$\begin{pmatrix} \triangle_1 \\ \triangle_2 \\ \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} \delta - \beta + n\Gamma \\ \beta - n\Gamma \\ \gamma \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \equiv \mathbf{z}_\lambda, \qquad \lambda \in \mathbb{R}.$$

Among these solution vectors $\mathbf{z}_\lambda$, we will find a vector satisfying all of the constraint inequalities needed below.

Now by Young's inequality (8.19) and (8.20), we have

$$\frac{w(x,y)}{v(u,t)} \leq \frac{W(x,y)}{V(u,t)},$$

and so by the sandwiching principle, Lemma 1, we have

$$N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \leq N_{p,q}^{(\alpha,\beta),(m,n)}(V,W). \tag{8.23}$$

Moreover, the weights V, W are product weights,

$$\begin{aligned} V(u,t) &= V_1(u) V_2(t) \text{ and } W(x,y) = W_1(x) W_2(y); \\ V_1(u) &= |u|^{\frac{\delta_1 m}{m+n}}, \quad V_2(t) = |t|^{\frac{\delta_2 n}{m+n}}, \\ W_1(x) &= \left(\frac{1}{|x|}\right)^{\frac{\gamma_1 m}{m+n}}, \quad W_2(y) = \left(\frac{1}{|y|}\right)^{\frac{\gamma_2 n}{m+n}}, \end{aligned}$$

where the 1-parameter weight pairs $(V_1(u), W_1(x))$ and $(V_2(t), W_2(y))$ each satisfy the hypotheses of Theorem 3 on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively for an appropriate choice of solution vector above.

Indeed, to see this, note that the first weight pair $(V_1(u), W_1(x)) = \left(|u|^{\triangle_1}, |x|^{-\Gamma_1}\right)$ on $\mathbb{R}^m$ satisfies the equality

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\Gamma_1 + \triangle_1)}{m} \tag{8.24}$$

by (8.21). We also claim the inequalities

$$q\Gamma_1 < m \text{ and } p'\triangle_1 < m \text{ and } \Gamma_1 + \triangle_1 \geq 0, \tag{8.25}$$

for an appropriate family of solution vectors $\mathbf{z}_\lambda$. The third inequality actually holds for all solution vectors $\mathbf{z}_\lambda$ since

$$\Gamma_1 + \triangle_1 = \gamma + \delta - \beta + n\Gamma \geq 0,$$

by (4.19). Thus the equality (8.24), and the inequalities (8.25), all hold for those solution vectors $\mathbf{z}_\lambda$ satisfying

$$\Gamma_1 < \frac{m}{q} \text{ and } \triangle_1 < \frac{m}{p'}; \tag{8.26}$$

$$\text{i.e} \quad \gamma + \lambda < \frac{m}{q} \text{ and } \delta - \beta + n\Gamma - \lambda < \frac{m}{p'};$$

$$\text{i.e.} \quad \delta - \beta + n\Gamma - \frac{m}{p'} < \lambda < \frac{m}{q} - \gamma.$$

The second weight pair $(V_2(t), W_2(y)) = \left(|t|^{\triangle_2}, |y|^{-\Gamma_2}\right)$ on $\mathbb{R}^n$ satisfies the equality

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\Gamma_2 + \triangle_2)}{n} \tag{8.27}$$

by (8.21). We also claim the inequalities

$$q\Gamma_2 < n \text{ and } p'\triangle_2 < n \text{ and } \Gamma_2 + \triangle_2 \geq 0, \tag{8.28}$$

for an appropriate family of solution vectors $\mathbf{z}_\lambda$. The third inequality actually holds for all solution vectors $\mathbf{z}_\lambda$ since

$$\Gamma_2 + \triangle_2 = \alpha + \beta - (m+n)\Gamma = \gamma + \delta \geq 0,$$

by (4.19). Thus the equality (8.27), and the inequalities (8.28), all hold for those solution vectors $\mathbf{z}_\lambda$ satisfying

$$\begin{aligned} &\Gamma_2 < \frac{n}{q} \text{ and } \triangle_2 < \frac{n}{p'}; \\ \text{i.e.}\quad &-\lambda < \frac{n}{q} \text{ and } \beta - n\Gamma + \lambda < \frac{n}{p'}; \\ \text{i.e.}\quad &-\frac{n}{q} < \lambda < \frac{n}{p'} - \beta + n\Gamma. \end{aligned} \tag{8.29}$$

In order to find λ satisfying (8.26) and (8.29) simultaneously, we must establish the four strict inequalities in

$$\max\left\{\delta - \beta + n\Gamma - \frac{m}{p'}, -\frac{n}{q}\right\} < \min\left\{\frac{m}{q} - \gamma, \frac{n}{p'} - \beta + n\Gamma\right\}.$$

Now two of these four strict inequalities follow from the local integrability of the weights $|(x,y)|^{-\gamma q}$ and $|(u,t)|^{-\gamma p'}$ on $\mathbb{R}^m \times \mathbb{R}^n$, namely

$$\begin{aligned} \delta - \beta + n\Gamma - \frac{m}{p'} &< \frac{n}{p'} - \beta + n\Gamma \text{ and } -\frac{n}{q} < \frac{m}{q} - \gamma; \\ \text{i.e. } \delta &< \frac{m+n}{p'} \text{ and } \gamma < \frac{m+n}{q}. \end{aligned}$$

The other two strict inequalities follow from the assumptions that $\alpha < m$ and $\beta < n$, namely

$$\begin{aligned} &\delta - \beta + n\Gamma - \frac{m}{p'} < \frac{m}{q} - \gamma \text{ and } -\frac{n}{q} < \frac{n}{p'} - \beta + n\Gamma; \\ \text{i.e.}\quad &\delta + \gamma < \beta - n\Gamma + m\left(\frac{1}{p'} + \frac{1}{q}\right) \text{ and } \beta < n\left(\frac{1}{p'} + \frac{1}{q}\right) + n\Gamma; \\ \text{i.e.}\quad &\delta + \gamma < \beta - n\Gamma + m(1-\Gamma) \text{ and } \beta < n(1-\Gamma) + n\Gamma; \\ \text{i.e.}\quad &(m+n)\Gamma + (\delta + \gamma) < m + \beta \text{ and } \beta < n; \\ \text{i.e.}\quad &\alpha + \beta < m + \beta \text{ and } \beta < n, \end{aligned}$$

where in the final line above we have used the power weight equality from the first line of (4.19).

Thus there does indeed exist a choice of $\gamma \in \mathbb{R}$ so that the equalities (8.24), (8.27) and inequalities (8.25), (8.28) all hold. It now follows from Theorem 13 that

$$\begin{aligned} N_{p,q}^{(\alpha,\beta),(m,n)}(V,W) &\leq N_{p,q}^{\alpha,m}(V_1,W_1)\; N_{p,q}^{\beta,n}(V_2,W_2) \\ &\leq C A_{p,q}^{\alpha,m}(V_1,W_1)\; A_{p,q}^{\beta,n}(V_2,W_2) < \infty, \end{aligned}$$

and combined with (8.23) this yields

$$N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) < \infty,$$

in the case $\gamma \geq 0$ and $\delta \geq 0$.

Case 2: Next we suppose that $\gamma < 0 < \delta$ and use the fourth line in (8.1). Let $\rho \equiv \gamma + \delta \geq 0$ and $\eta \equiv -\gamma > 0$. Then by Young's inequality (8.19), the weight pairs

$$\begin{aligned} (V(u,t), W(x,y)) &\equiv \left(|u|^{\frac{\rho_1 m}{m+n}+\eta_1} |t|^{\frac{\rho_1 n}{m+n}}, |x|^{\eta}\right), \\ (V'(u,t), W'(x,y)) &\equiv \left(|u|^{\frac{\rho_2 m}{m+n}} |t|^{\frac{\rho_2 n}{m+n}+\eta_2}, |y|^{\eta}\right), \end{aligned}$$

where $\rho_j + \eta_j = \rho + \eta = \delta > 0$ for $j = 1, 2$, satisfy

$$\begin{aligned}
\frac{w(x,y)}{v(u,t)} &= \frac{\left(|x|^2+|y|^2\right)^{-\frac{\gamma}{2}}}{\left(|u|^2+|t|^2\right)^{\frac{\delta}{2}}} = \frac{\left(|x|^2+|y|^2\right)^{\frac{\eta}{2}}}{\left(|u|^2+|t|^2\right)^{\frac{\rho+\eta}{2}}} \\
&\lesssim \frac{|x|^\eta+|y|^\eta}{\left(|u|^2+|t|^2\right)^{\frac{\rho_1}{2}}\left(|u|^2+|t|^2\right)^{\frac{\eta_1}{2}}} + \frac{|x|^\eta+|y|^\eta}{\left(|u|^2+|t|^2\right)^{\frac{\rho_2}{2}}\left(|u|^2+|t|^2\right)^{\frac{\eta_2}{2}}} \\
&\lesssim \frac{|x|^\eta}{\left(|u|^{\frac{m}{m+n}}|t|^{\frac{n}{m+n}}\right)^{\rho_1}|u|^{\eta_1}} + \frac{|y|^\eta}{\left(|u|^{\frac{m}{m+n}}|t|^{\frac{n}{m+n}}\right)^{\rho_2}|t|^{\eta_2}} = \frac{W(x,y)}{V(u,t)} + \frac{W'(x,y)}{V'(u,t)},
\end{aligned}$$

and so by the sandwiching principle, Lemma 1, we have

$$N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \le N_{p,q}^{(\alpha,\beta),(m,n)}(V,W) + N_{p,q}^{(\alpha,\beta),(m,n)}(V',W').$$

Moreover, the weights V, V', W, W' are product weights,

$$\begin{aligned}
V(u,t) &= V_1(u)\,V_2(t) \text{ and } V'(u,t) = V_1'(u)\,V_2'(t), \\
W(x,y) &= W_1(x)\,W_2(y) \text{ and } W'(x,y) = W_1'(x)\,W_2'(y),
\end{aligned}$$

where

$$\left\{\begin{array}{cccc} V_1(u) = |u|^{\frac{\rho_1 m}{m+n}+\eta_1} & V_2(t) = |t|^{\frac{\rho_1 n}{m+n}} & V_1'(u) = |u|^{\frac{\rho_2 m}{m+n}} & V_2'(t) = |t|^{\frac{\rho_2 n}{m+n}+\eta_2} \\ W_1(x) = |x|^\eta & W_2(y) = 1 & W_1'(x) = 1 & W_2'(y) = |y|^\eta \end{array}\right. ,$$

and where the 1-parameter weight pairs $(V_1(u), W_1(x)), (V_1'(u), W_1'(x))$ on $\mathbb{R}^m$ and $(V_2(t), W_2(y)), (V_2'(t), W_2'(y))$ on $\mathbb{R}^n$ each satisfy the hypotheses of Theorem 3 provided we choose ρ_1 and η_1 to satisfy

$$\begin{aligned}
\frac{\alpha - \left(-\eta + \frac{\rho_1 m}{m+n} + \eta_1\right)}{m} &= \Gamma = \frac{\alpha+\beta-\rho}{m+n}, \\
\text{and } \frac{\beta - \left(-0 + \frac{\rho_1 n}{m+n}\right)}{n} &= \Gamma = \frac{\alpha+\beta-\rho}{m+n},
\end{aligned} \tag{8.30}$$

i.e.

$$\begin{aligned}
\frac{\alpha}{m} - \frac{\rho_1}{m+n} + \frac{\eta-\eta_1}{m} &= \frac{\alpha+\beta-\rho}{m+n} = \frac{\beta}{n} - \frac{\rho_1}{m+n}; \\
\frac{\alpha}{m} + \frac{\eta-\eta_1}{m} &= \frac{\alpha+\beta+\rho_1-\rho}{m+n} = \frac{\beta}{n}; \\
\frac{\beta}{n} - \frac{\alpha}{m} &= \frac{\eta-\eta_1}{m} \text{ and } \frac{\alpha+\beta}{m+n} = \frac{\beta}{n} + \frac{\rho-\rho_1}{m+n}; \\
\frac{\eta-\eta_1}{m} &= \frac{\beta}{n} - \frac{\alpha}{m} \text{ and } \frac{\rho-\rho_1}{m+n} = \frac{\alpha+\beta}{m+n} - \frac{\beta}{n}; \\
\frac{\eta_1}{m} &= \frac{\alpha+\eta}{m} - \frac{\beta}{n} \text{ and } \frac{\rho_1}{m+n} = \frac{\rho-(\alpha+\beta)}{m+n} + \frac{\beta}{n},
\end{aligned}$$

so that

$$\eta_1 = \alpha + \eta - \frac{m}{n}\beta \text{ and } \rho_1 = \rho - (\alpha+\beta) + \frac{m+n}{n}\beta, \tag{8.31}$$

and where ρ_2 and η_2 will be chosen below. Once we have established the appropriate hypotheses of Theorem 3, Theorem 13 will show that

$$N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) < \infty,$$

in the case $\gamma < 0 < \delta$.

To show that these four weight pairs satisfy the hypotheses of Theorem 3, we first note that

$$\rho_1 = (m+n)\left[\frac{(\gamma+\delta)-(\alpha+\beta)}{m+n} + \frac{\beta}{n}\right] = (m+n)\left(\frac{\beta}{n} - \Gamma\right) \ge 0 \tag{8.32}$$

by the power weight equality in (4.19), and the third line in (4.19), and also note that

$$\begin{aligned}\rho_1+\eta_1 &= (m+n)\left\{\frac{\rho-(\alpha+\beta)}{m+n}+\frac{\beta}{n}\right\}+m\left\{\frac{\alpha+\eta}{m}-\frac{\beta}{n}\right\}\\ &= \rho-(\alpha+\beta)+\frac{m+n}{n}\beta+(\alpha+\eta)-\frac{m}{n}\beta=\rho+\eta.\end{aligned}$$

Next, we verify that the first weight pair $(V_1(u),W_1(x))=\left(|u|^{\frac{\rho_1 m}{m+n}+\eta_1},|x|^{\eta}\right)$ on $\mathbb{R}^m$ satisfies the 1-parameter power weight equality

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha-\left(-\eta+\frac{\rho_1 m}{m+n}+\eta_1\right)}{m}, \tag{8.33}$$

as well as the 1-parameter constraint inequalities

$$q(-\eta)<m \text{ and } p'\left(\frac{\rho_1 m}{m+n}+\eta_1\right)<m \text{ and } -\eta+\frac{\rho_1 m}{m+n}+\eta_1\geq 0. \tag{8.34}$$

The equality (8.33) follows immediately from (8.31), the first line in (4.19), and (8.30). The first inequality in (8.34) is trivial, and the third inequality in (8.34) is

$$-\eta+\frac{\rho_1 m}{m+n}+\eta_1\geq 0,$$

which follows from (8.31):

$$\begin{aligned}-\eta+\frac{m}{m+n}\rho_1+\eta_1 &= -\eta+\frac{m}{m+n}\left[\rho-(\alpha+\beta)+\frac{m+n}{n}\beta\right]+\alpha+\eta-\frac{m}{n}\beta\\ &= -\eta+\frac{m}{m+n}(\rho-(\alpha+\beta))+\frac{m}{n}\beta+\alpha+\eta-\frac{m}{n}\beta\\ &= \frac{m}{m+n}(\rho-(\alpha+\beta))+\alpha\\ &= m\left(\frac{\rho-(\alpha+\beta)}{m+n}+\frac{\alpha}{m}\right)=m\left(\frac{\alpha}{m}-\Gamma\right)\geq 0.\end{aligned} \tag{8.35}$$

The second inequality in (8.34) is

$$\frac{1}{m}\left(\frac{\rho_1 m}{m+n}+\eta_1\right)<\frac{1}{p'}, \tag{8.36}$$

which, using $\gamma=-\eta$ with the equality $-\eta+\frac{m}{m+n}\rho_1+\eta_1=m\left(\frac{\alpha}{m}-\Gamma\right)=\alpha-m\Gamma$ just proved above in (8.35), is equivalent to

$$\begin{aligned}&\frac{1}{m}(\alpha-m\Gamma+\eta)<\frac{1}{p'};\\ \Longleftrightarrow\ &\frac{\alpha+\eta}{m}-\Gamma<\frac{1}{p'};\\ \Longleftrightarrow\ &\frac{\alpha}{m}<\Gamma+\frac{\gamma}{m}+\frac{1}{p'};\\ \Longleftrightarrow\ &\frac{\alpha}{m}-\frac{1}{q'}<\frac{\gamma}{m}.\end{aligned}$$

Next we note that the weight pair $(V_2(t),W_2(y))=\left(|t|^{\frac{\rho_1 n}{m+n}},1\right)$ on $\mathbb{R}^n$ satisfies the 1-parameter power weight equality

$$\frac{1}{p}-\frac{1}{q}=\frac{\beta-\left(-0+\frac{\rho_1 m}{m+n}\right)}{n},$$

and the 1-parameter constraint inequalities,

$$q(-0)<n \text{ and } p'\left(\frac{\rho_1 n}{m+n}\right)<n \text{ and } -0+\frac{\rho_1 m}{m+n}\geq 0.$$

Indeed, for the equality we use (8.31), the first line in (4.19), and (8.30). The first of the constraint inequalities is trivial and the third constraint inequality follows from (8.32). The second of the constraint inequalities, namely $p'\left(\frac{\rho_1 n}{m+n}\right) < n$, is equivalent to

$$\begin{aligned} &\left(\frac{n}{m+n}\right)(m+n)\left(\frac{\beta}{n}-\Gamma\right) < \frac{n}{p'}; \\ \text{i.e.} \qquad &\frac{\beta}{n} < \Gamma + \frac{1}{p'}. \end{aligned}$$

However from the third line in (4.19) and the assumption that $\gamma < 0$ we have

$$\begin{aligned} \frac{\beta}{n} &\leq \Gamma + \frac{\gamma+\delta}{n} - \frac{\left(\delta - \frac{n}{p'}\right)_+}{n} \\ &= \Gamma + \frac{\gamma}{n} + \frac{1}{p'} + \frac{\delta - \frac{n}{p'}}{n} - \left(\frac{\delta}{n} - \frac{1}{p'}\right)_+ \\ &= \Gamma + \frac{\gamma}{n} + \frac{1}{p'} - \left(\frac{\delta}{n} - \frac{1}{p'}\right)_- \\ &\leq \Gamma + \frac{\gamma}{n} + \frac{1}{p'} < \Gamma + \frac{1}{p'}, \end{aligned}$$

and this time there is no exceptional endpoint case. As indicated above, this completes the proof in the case $\gamma < 0 < \delta$.

The same arguments apply to the weight pair (V', W'), which we now sketch briefly. The weight pairs $(V_2'(t), W_2'(y)) = \left(|t|^{\frac{\rho_2 n}{m+n}+\eta_2}, |y|^{\eta}\right)$ and $(V_1'(u), W_1'(x)) = \left(|u|^{\frac{\rho_2 m}{m+n}}, 1\right)$ each satisfy the hypotheses of Theorem 3 provided we choose ρ_2 and η_2 to satisfy

$$\begin{aligned} \frac{\beta - \left(-\eta + \frac{\rho_2 n}{m+n} + \eta_2\right)}{n} &= \Gamma = \frac{\alpha+\beta-\rho}{m+n}, \\ \text{and } \frac{\alpha - \left(-0 + \frac{\rho_2 m}{m+n}\right)}{m} &= \Gamma = \frac{\alpha+\beta-\rho}{m+n}, \end{aligned} \tag{8.37}$$

i.e.

$$\begin{aligned} \frac{\alpha}{m} - \frac{\rho_2}{m+n} &= \frac{\alpha+\beta-\rho}{m+n} = \frac{\beta}{n} - \frac{\rho_2}{m+n} + \frac{\eta-\eta_2}{n}; \\ \frac{\alpha}{m} &= \frac{\alpha+\beta+\rho_2-\rho}{m+n} = \frac{\beta}{n} + \frac{\eta-\eta_2}{n}; \\ \frac{\alpha}{m} - \frac{\beta}{n} &= \frac{\eta-\eta_2}{n} \text{ and } \frac{\alpha+\beta}{m+n} = \frac{\alpha}{m} + \frac{\rho-\rho_2}{m+n}; \\ \frac{\eta_2}{n} &= \frac{\beta+\eta}{n} - \frac{\alpha}{m} \text{ and } \frac{\rho_2}{m+n} = \frac{\rho-(\alpha+\beta)}{m+n} + \frac{\alpha}{m}; \end{aligned}$$

so that

$$\eta_2 = \beta + \eta - \frac{n}{m}\alpha \text{ and } \rho_2 = \rho - (\alpha+\beta) + \frac{m+n}{m}\alpha. \tag{8.38}$$

All of the constraint inequalities hold by arguments similar to those above, except of course in the analogous exceptional endpoint case, and by way of example we treat just the second of the constraint inequalities for the weight pair that gives rise to the exceptional endpoint case. The second constraint inequality for the weight pair $(V_2'(t), W_2'(y)) = \left(|t|^{\frac{\rho_2 m}{m+n}+\eta_2}, |y|^{\eta}\right)$ on $\mathbb{R}^n$ is

$$\frac{1}{n}\left(\frac{\rho_2 m}{m+n} + \eta_2\right) < \frac{1}{p'}. \tag{8.39}$$

Using $\gamma = -\eta$ with the equality

$$\begin{aligned} & -\eta + \frac{n}{m+n}\rho_2 + \eta_2 \\ = \quad & -\eta + \frac{n}{m+n}\left[\rho - (\alpha+\beta) + \frac{m+n}{m}\alpha\right] + \beta + \eta - \frac{n}{m}\alpha \\ = \quad & -\eta + \frac{n}{m+n}\left(\rho - (\alpha+\beta)\right) + \frac{n}{m}\alpha + \beta + \eta - \frac{n}{m}\alpha \\ = \quad & \frac{n}{m+n}\left(\rho - (\alpha+\beta)\right) + \beta \\ = \quad & n\left(\frac{\rho - (\alpha+\beta)}{m+n} + \frac{\beta}{n}\right) = n\left(\frac{\beta}{n} - \Gamma\right), \end{aligned}$$

we see that (8.39) is equivalent to

$$\begin{aligned} & \frac{1}{n}\left(n\left(\frac{\beta}{n} - \Gamma\right) + \eta\right) < \frac{1}{p'} \\ \iff \quad & \frac{\beta}{n} < \Gamma + \frac{1}{p'} + \frac{\gamma}{n} \\ \iff \quad & \frac{\beta}{n} - \frac{1}{q'} < \frac{\gamma}{n}. \end{aligned}$$

Case 3: The case $\delta < 0 < \gamma$ is handled similarly using the third line in (8.1).

8.2. **The exceptional cases $\alpha = m$ or $\beta = n$.** Here we consider the two cases where $\alpha = m$ or $\beta = n$. We first show that the power weight norm inequality (4.12) holds when $\alpha = m$ and $p \leq q$ and $A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma) < \infty$. To see this, we note that from the first line in (**??**) that

$$\begin{aligned} 0 \quad & < \quad \beta < n, \\ \frac{m}{q} \quad & < \quad \gamma < \frac{m+n}{q}, \\ \frac{m}{p'} \quad & < \quad \delta < \frac{m+n}{p'}. \end{aligned}$$

Then we compute that

$$I_{m,\beta}^{m,n} f(x,y) = \int\int_{\mathbb{R}^m \times \mathbb{R}^n} |y-t|^{\beta - n} f(u,t)\, du dt = \int_{\mathbb{R}^n} |y-t|^{\beta-n} F(t)\, dt = I_\beta^n F(y),$$

where $F(t) \equiv \int_{\mathbb{R}^m} f(u,t)\, du$. Thus we have

$$\begin{aligned} & \int\int_{\mathbb{R}^m \times \mathbb{R}^n} \left|I_{m,\beta}^{m,n} f(x,y)\right|^q \ |(x,y)|^{-\gamma q}\, dx dy \\ = \quad & \int_{\mathbb{R}^n} \left|I_\beta^n F(y)\right|^q \left(\int_{\mathbb{R}^m} |(x,y)|^{-\gamma q}\, dx\right) dy \\ \approx \quad & \int_{\mathbb{R}^n} \left|I_\beta^n F(y)\right|^q |y|^{m-\gamma q}\, dy, \end{aligned}$$

since

$$\int_{\mathbb{R}^m} |(x,y)|^{-\gamma q}\, dx \approx \int_{\mathbb{R}^m} |(|x|+|y|)|^{-\gamma q}\, dx = |y|^{m-\gamma q} \int_{\mathbb{R}^m} \left|\left(\frac{|x|}{|y|} + 1\right)\right|^{-\gamma q} d\left(\frac{x}{|y|}\right) \approx |y|^{m-\gamma q}$$

for $m-\gamma q<0$. We also have

$$\begin{aligned}\int_{\mathbb{R}^n}|F(t)|^p|t|^{\left(\delta-\frac{m}{p'}\right)p}dt &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^m}f(u,t)\,du\right|^p|t|^{\left(\delta-\frac{m}{p'}\right)p}dt\\ &\leq \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}|f(u,t)|^p(|u|+|t|)^{\delta p}du\right\}\left\{\int_{\mathbb{R}^m}(|u|+|t|)^{-\delta p'}du\right\}^{p-1}|t|^{\left(\delta-\frac{m}{p'}\right)p}dt\\ &\approx \int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}|f(u,t)|^p(|u|+|t|)^{\delta p}du\right\}\left\{|t|^{m-\delta p'}\right\}^{p-1}|t|^{\left(\delta-\frac{m}{p'}\right)p}dt\\ &= \int\int_{\mathbb{R}^m\times\mathbb{R}^n}|f(u,t)|^p(|u|+|t|)^{\delta p}dudt,\end{aligned}$$

since

$$\int_{\mathbb{R}^m}(|u|+|t|)^{-\delta p'}du\approx|t|^{m-\delta p'}$$

for $m-\delta p'<0$. Thus we conclude that (4.12) holds provided we have the 1-parameter power weight norm inequality

$$\left(\int_{\mathbb{R}^n}\left|I_\beta^nF(y)\right|^q|y|^{-\left(\gamma-\frac{m}{q}\right)q}dy\right)^{\frac{1}{q}}\lesssim\left(\int_{\mathbb{R}^n}|F(t)|^p|t|^{\left(\delta-\frac{m}{p'}\right)p}dt\right)^{\frac{1}{p}}.$$

But this inequality holds by Theorem 3 since the 1-parameter power weight equality holds,

$$\begin{aligned}\beta-\left(\gamma-\frac{m}{q}\right)-\left(\delta-\frac{m}{p'}\right) &= \beta-\gamma-\delta+m\left(\frac{1}{q}+\frac{1}{p'}\right)\\ &= \alpha+\beta-\gamma-\delta+m\left(\frac{1}{q}+\frac{1}{p'}-1\right)\\ &= (m+n)\Gamma-m\Gamma=n\Gamma,\end{aligned}$$

and each of the following four constraint inequalities holds,

$$\begin{aligned}&0<\beta<n,\\ &\left(\gamma-\frac{m}{q}\right)<\frac{n}{q},\\ &\left(\delta-\frac{m}{p'}\right)<\frac{n}{p'},\\ &\left(\gamma-\frac{m}{q}\right)+\left(\delta-\frac{m}{p'}\right)\geq 0.\end{aligned}$$

A similar argument shows that (4.12) holds when $\beta=n$ and $p\leq q$ and $A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta,w_\gamma)<\infty$.

9. Proof of Theorem 11

Define the eccentricity $\varkappa(R)$ of a rectangle $R=I\times J$ to be $\varkappa(R)=\frac{\ell(I)}{\ell(J)}$. For $j\in\mathbb{Z}$ define the conical operator $\triangle_jI_{\alpha,\beta}$ acting on a measure μ by

$$\triangle_jI_{\alpha,\beta}\mu(x,y)\equiv\int\int_{\mathbb{S}_j+(x,y)}\left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta}d\mu(u,t),$$

where $\mathbb{S}_j\equiv\left\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^n:2^{-j-1}\leq\frac{|y|}{|x|}<2^{-j+1}\right\}$ is a cone with aperature roughly 2^{-j} and slope roughly 2^{-j}.

Lemma 4. *Suppose $1<p<q<\infty$, $0<\alpha,\beta<1$ and that both σ and ω are rectangle doubling, and that the reverse doubling exponent ε for σ satisfies*

$$1-\varepsilon<\frac{\alpha}{m}=\frac{\beta}{n}.$$

For $\frac{1}{p}-\frac{1}{q}<\frac{\alpha}{m}=\frac{\beta}{n}$ *we have*

$$\left(\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\left|\triangle_j I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)(x,y)\right|^q d\omega(x,y)\right)^{\frac{1}{q}}\lesssim 2^{-\varepsilon'|j-k|}\left(\iint_R d\sigma\right)^{\frac{1}{p}}, \tag{9.1}$$

for all rectangles R *with eccentricity* $\varkappa(R)=2^{-k}$*, and where* $\varepsilon'>0$ *depends on* $p,q,\alpha,\beta,\varepsilon$*.*

Proof. We will prove the special case when $\ell(I)=1$ and $\varkappa(R)=1$, i.e. R is a square of side length 1. The general case is similar. We now place the origin so that R is the block $B_0=[1,2]^m\times[1,2]^n$.

We first consider the region $\mathcal{R}_1\equiv\left\{(x,y):|x|\leq\frac{1}{4}|y|\right\}$. In this region the sum over $j<0$ is easy. So we consider $j>0$. To see that (9.1) holds with integration on the left restricted to $\mathcal{R}_1$, we begin by noting that

$$\begin{aligned}\triangle_j I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)(x,y)&=\int\int_{R\cap\{\mathbb{S}_j+(x,y)\}}\left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta}d\sigma(u,t)\\&=\sum_{r=1}^{2^j}\left[\iint_{R(r)}\left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta}d\sigma(u,t)\right]\mathbf{1}_{R^*(r)}(x,y),\end{aligned}$$

where the tiles $R(r)$ are rectangles of size 1 by 2^{-j} and the tiles $R^*(r)$ are slightly enlarged reflections of the $R(r)$ centered on the y-axis. Now we continue by computing

$$\begin{aligned}&\iint_{\mathcal{R}_1}\left|\triangle_j I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)(x,y)\right|^q d\omega(x,y)\\&=\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\left\{\sum_{r=1}^{2^j}\left[\iint_{R(r)}\left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta}d\sigma(u,t)\right]\mathbf{1}_{R^*(r)}(x,y)\right\}^q d\omega(x,y)\\&\approx\sum_{r=1}^{2^j}\int\int_{R^*(r)}\left\{\iint_{R(r)}\left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta}d\sigma(u,t)\right\}^q d\omega(x,y),\end{aligned}$$

where matters have been reduced to the diagonal terms since $R^*(r)\cap R^*(s)=\emptyset$ unless $|r-s|\leq c_0$ (this follows easily from the fact that the tiles $R(r)$ are pairwise disjoint in r). Now we apply Hölder's inequality to obtain

$$\begin{aligned}&\int\int_{\mathbb{R}^m\times\mathbb{R}^n}\left|I_{\alpha,\beta}\left(\mathbf{1}_{B_0}\sigma\right)\right|^q w^q \qquad (9.2)\\&\lesssim\sum_{r=1}^{2^j}\int\int_{R^*(r)}\left\{\iint_{R(r)}d\sigma\right\}^{\frac{q}{p}}\left\{\iint_{R(r)}\left(\frac{1}{|x-u|}\right)^{(m-\alpha)p'}\left(\frac{1}{|y-t|}\right)^{(n-\beta)p'}d\sigma(u,t)\right\}^{\frac{q}{p'}}w(x,y)^q\,dxdy\\&\lesssim\left(A_{p,q}^{\alpha,\beta}\right)^q\sum_{r=1}^{2^j}\left\{\iint_{R(r)}d\sigma\right\}^{\frac{q}{p}}=\left(A_{p,q}^{\alpha,\beta}\right)^q\sum_{r=1}^{2^j}\left\{\iint_{R(r)}d\sigma\right\}^{\frac{q}{p}-1}\left(\iint_{R(r)}d\sigma\right)\\&\lesssim\left(A_{p,q}^{\alpha,\beta}\right)^q\sum_{r=1}^{2^j}\left\{2^{-\varepsilon j}\iint_R d\sigma\right\}^{\frac{q}{p}-1}\left(\iint_{R(r)}d\sigma\right)=2^{-\varepsilon' j}\left(\iint_R d\sigma\right)^{\frac{q}{p}}.\end{aligned}$$

Now we consider the region $\mathcal{R}_2 \equiv \left\{(x,y) : \frac{1}{4}|y| \leq |x| \leq 4|y|\right\}$. Here we must perform an additional calculation involving the intersection of the tile $R(r)$ and the cone $\mathbb{S}_j + (x,y)$:

$$\begin{aligned}&\iint_{\mathcal{R}_2} \left|\triangle_j I_{\alpha,\beta}\left(\mathbf{1}_{R(r)\cap[\mathbb{S}_j+(x,y)]}\sigma\right)(x,y)\right|^q d\omega(x,y)\\ =\ &\iint_{\mathcal{R}_2} \left\{\sum_{r=1}^{2^j}\left[\int\int_{R(r)\cap[\mathbb{S}_j+(x,y)]} \left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta} d\sigma(u,t)\right]\mathbf{1}_{R^*(r)}(x,y)\right\}^q d\omega(x,y),\end{aligned}$$

where now $R^*(r)$ can overlap $R(r)$ considerably. However, for each fixed $(x,y) \in R(r)$ we further decompose

$$R(r) \cap [\mathbb{S}_j + (x,y)] = \bigcup_{\ell} R_\ell(r)$$

where the widths of the $R_\ell(r)$ form a geometric sequence that approaches 0 as $R_\ell(r)$ approaches (x,y), and the dependence of $R_\ell(r)$ on (x,y) is suppressed. Moreover, the tiles $R_\ell(r)$ are roughly rectangles of dimension $2^{-\ell} \times 2^{-\ell-j}$, and so

$$\begin{aligned}&\int\int_{R(r)\cap[\mathbb{S}_j+(x,y)]} \left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta} d\sigma(u,t) \lesssim \sum_{\ell=1}^{\infty}\iint_{R_\ell(r)} \left(\frac{1}{2^{-\ell}}\right)^{m-\alpha}\left(\frac{1}{2^{-\ell-j}}\right)^{n-\beta} d\sigma(u,t)\\ \approx\ & 2^{j(n-\beta)}\sum_{\ell=1}^{\infty} 2^{\ell(m-\alpha+n-\beta)}\left|R_\ell(r)\right|_\sigma \lesssim 2^{j(n-\beta)}\sum_{\ell=1}^{\infty} 2^{\ell(m-\alpha+n-\beta)}\left(2^{-\varepsilon(m\ell+n\ell+jn)}|R|_\sigma\right)\\ \lesssim\ & 2^{j(n-\beta-\varepsilon n)}\sum_{\ell=1}^{\infty} 2^{\ell(m-\alpha+n-\beta-(m+n)\varepsilon)}|R|_\sigma \approx 2^{jn\left(1-\frac{\beta}{n}-\varepsilon\right)}|R|_\sigma\end{aligned}$$

provided $1-\varepsilon < \frac{\alpha}{m} = \frac{\beta}{n}$. Thus we have

$$\begin{aligned}&\iint_{\mathcal{R}_2} \left|\triangle_j I_{\alpha,\beta}(\mathbf{1}_R\sigma)\right|^q d\omega\\ =\ &\iint_{\mathcal{R}_2} \left\{\sum_{r=1}^{2^j}\sum_{\ell=1}^{N}\left[\int\int_{R_\ell(r)\cap[\mathbb{S}_j+(x,y)]} \left(\frac{1}{|x-u|}\right)^{m-\alpha}\left(\frac{1}{|y-t|}\right)^{n-\beta} d\sigma(u,t)\right]\mathbf{1}_{R_\ell^*(r)}(x,y)\right\}^q d\omega(x,y)\\ \lesssim\ &\iint_{\mathcal{R}_2} \left\{\sum_{r=1}^{2^j}\sum_{\ell=1}^{N}\left[2^{jn\left(1-\frac{\beta}{n}-\varepsilon\right)}|R|_\sigma\right]\mathbf{1}_{R_\ell^*(r)}(x,y)\right\}^q d\omega(x,y)\\ \lesssim\ &\sum_{r=1}^{2^j}\iint_{\mathcal{R}_2} \left\{\sum_{\ell=1}^{N}\left[2^{jn\left(1-\frac{\beta}{n}-\varepsilon\right)}|R|_\sigma\right]\mathbf{1}_{R_\ell^*(r)}(x,y)\right\}^q d\omega(x,y)\\ \approx\ &\sum_{r=1}^{2^j}\sum_{\ell=1}^{N}\iint_{\mathcal{R}_2} \left\{\left[2^{jn\left(1-\frac{\beta}{n}-\varepsilon\right)}|R|_\sigma\right]\mathbf{1}_{R_\ell^*(r)}(x,y)\right\}^q d\omega(x,y),\end{aligned}$$

since the $R_\ell^*(r)$ are essentially pairwise disjoint in both r and ℓ (there is also decay in the kernel in the parameter ℓ). Now we apply Hölder's inequality and continue as in (9.2) above.

Region $\mathcal{R}_3 \equiv \{(x,y) : |x| \leq 4|y|\}$ is handled symmetrically to Region $\mathcal{R}_1$, and this completes the proof of Lemma 4. □

Now we can easily obtain the testing condition.

Corollary 3. *Suppose $1 < p < q < \infty$, $0 < \alpha, \beta < 1$ and that both σ and ω are rectangle doubling, and that the reverse doubling exponent ε for σ satisfies*

$$1-\varepsilon < \frac{\alpha}{m} = \frac{\beta}{n}.$$

For $\frac{1}{p}-\frac{1}{q}<\frac{\alpha}{m}=\frac{\beta}{n}$ we have the testing condition

$$\left(\int\limits_{\mathbb{R}^m}\int\limits_{\times\mathbb{R}^n}\left|I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)\right|^q d\omega\right)^{\frac{1}{q}}\lesssim\mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}\left(\sigma,\omega\right)\left(\iint\limits_R d\sigma\right)^{\frac{1}{p}},\qquad \text{for all } R=I\times J,$$

as well as the dual testing condition in which the roles of the measures σ and ω are reversed, and the exponents p,q are replaced with q',p' respectively.

Proof. We prove the testing condition, and leave the dual testing condition to the reader. Suppose that R has eccentricity $\varkappa(R)=2^{-k}$. Then from Minkowski's inequality and Lemma 4 we have

$$\begin{aligned}\left(\int\limits_{\mathbb{R}^m}\int\limits_{\times\mathbb{R}^n}\left|I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)\right|^q d\omega\right)^{\frac{1}{q}} &= \left(\int\limits_{\mathbb{R}^m}\int\limits_{\times\mathbb{R}^n}\left|\sum_{i\in\mathbb{Z}}I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)\right|^q d\omega\right)^{\frac{1}{q}}\\ &\leq \sum_{i\in\mathbb{Z}}\left(\int\limits_{\mathbb{R}^m}\int\limits_{\times\mathbb{R}^n}\left|I_{\alpha,\beta}\left(\mathbf{1}_R\sigma\right)\right|^q d\omega\right)^{\frac{1}{q}}\\ &\leq \sum_{i\in\mathbb{Z}}2^{-\varepsilon'|j-k|}\left(\iint\limits_R d\sigma\right)^{\frac{1}{p}}=C_{\varepsilon'}\left(\iint\limits_R d\sigma\right)^{\frac{1}{p}}.\end{aligned}$$

□

10. Appendix

10.1. **Sharpness in the Muckenhoupt-Wheeden theorem.** Here we give a sharp form of the Muckenhoupt-Wheeden Theorem 1.

Theorem 14. *Let $0<\alpha<m$, $1<p,q<\infty$, and let $w(x)$ be a nonnegative weight on $\mathbb{R}^m$. Then*

$$\left\{\int_{\mathbb{R}^m}I_\alpha^m f(x)^q\ w(x)^q\ dx\right\}^{\frac{1}{q}}\leq N_{p,q}^{\alpha,m}(w)\left\{\int_{\mathbb{R}^m}f(x)^p\ w(x)^p\ dx\right\}^{\frac{1}{p}}\tag{10.1}$$

for all $f\geq 0$ and $N_{p,q}^{\alpha,m}(w)<\infty$ if and only if *both*

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{m},\tag{10.2}$$

and

$$A_{p,q}(w)\equiv\sup_{cubes\ I\subset\mathbb{R}^m}\left(\frac{1}{|I|}\int_I w(x)^q\ dx\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I w(x)^{-p'}\ dx\right)^{\frac{1}{p'}}<\infty.$$

Proof. Given Theorem 1, it remains only to prove that (10.2) is necessary for (10.1). To see this, we apply Hölder's inequality with dual exponents $\frac{p'+1}{p'}$ and $p'+1$ to obtain

$$\begin{aligned}1 &= \left\{\frac{1}{|I|}\int_I w^{\frac{p'}{p'+1}}w^{-\frac{p'}{p'+1}}\right\}^{\frac{p'+1}{p'}}\\ &\leq \left\{\left(\frac{1}{|I|}\int_I w\right)^{\frac{p'}{p'+1}}\left(\frac{1}{|I|}\int_I w^{-p'}\right)^{\frac{1}{p'+1}}\right\}^{\frac{p'+1}{p'}}\\ &= \left(\frac{1}{|I|}\int_I w\right)\left(\frac{1}{|I|}\int_I w^{-p'}\right)^{\frac{1}{p'}}\leq\left(\frac{1}{|I|}\int_I w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I w^{-p'}\right)^{\frac{1}{p'}}.\end{aligned}$$

Then we conclude that

$$|I|^{\frac{\alpha}{m}-1+\frac{1}{q}+\frac{1}{p'}}\leq|I|^{\frac{\alpha}{m}-1+\frac{1}{q}+\frac{1}{p'}}\left(\frac{1}{|I|}\int_I w^q\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I w^{-p'}\right)^{\frac{1}{p'}}\leq A_{p,q}^{\alpha,m}(w)\leq N_{p,q}^{\alpha,m}(w)$$

for all cubes I, which implies $\frac{\alpha}{m}-1+\frac{1}{q}+\frac{1}{p'}=0$ as required. □

10.2. **Sharpness in the Stein-Weiss theorem.** Here we give a sharp form of the Stein-Weiss Theorem 3.

Theorem 15. *Let*

$$\begin{aligned} -\infty &< \alpha,\beta,\gamma,\delta<\infty, \\ 1 &< p,q<\infty, \\ m &\in \mathbb{N}. \end{aligned}$$

Then the power weighted norm inequality

$$\left\{\int_{\mathbb{R}^m} I_\alpha^m f(x)^q \ |x|^{-\gamma q} \ dx\right\}^{\frac{1}{q}} \leq N_{p,q}(w,v)\left\{\int_{\mathbb{R}^m} f(x)^p \ |x|^{\delta p} \ dx\right\}^{\frac{1}{p}}, \qquad \text{for all } f\geq 0, \tag{10.3}$$

holds if and only if *the power weight equality*

$$\frac{1}{p}-\frac{1}{q}=\frac{\alpha-(\gamma+\delta)}{m} \tag{10.4}$$

holds along with the constraint inequalities,

$$0<\alpha<m, \tag{10.5}$$

$$p\leq q, \tag{10.6}$$

$$q\gamma<m, \tag{10.7}$$

$$p'\delta<m, \tag{10.8}$$

and

$$\gamma+\delta\geq 0. \tag{10.9}$$

Proof. The sufficiency of these conditions for (10.3) is the classical theorem of Stein and Weiss, so we turn to proving their necessity. The required local integrability of the power weights $|x|^{-\gamma q}$ and $|x|^{-\delta p'}$ shows that (10.7) and (10.8) hold. The kernel $|x-y|^{\alpha-m}$ of the convolution operator must be locally integrable on $\mathbb{R}^m$ and this implies that $0<\alpha$. Using this, we can now use the argument in the proof of Proposition **??** to prove that finiteness of the Muckenhoupt characteristic $A_{p,q}(w,v)$ is necessary, and this in turn implies both (10.9) and the power weight equality (10.4) just as in the proof of Theorem 9 above for the 2-parameter case. From (10.4), (10.7) and (10.8) we now obtain

$$\alpha=m\left(\frac{1}{p}-\frac{1}{q}\right)+(\gamma+\delta)<m\left(\frac{1}{p}-\frac{1}{q}\right)+\left(\frac{m}{q}+\frac{m}{p'}\right)=m,$$

which completes the proof that (10.5) holds.

Finally we turn to proving (10.6). Let $f(y)=f(s)$, $s=|y|$, be a radial function on $\mathbb{R}^m$. Then $I_\alpha f(x)=I_\alpha f(r)$, $r=|x|$, is also radial and

$$I_\alpha f(x)=\int_{\mathbb{R}^m}|x-y|^{\alpha-m}f(y)\,dy\gtrsim\int_{|y|\leq|x|}|x|^{\alpha-m}f(y)\,dy=r^{\alpha-m}\int_0^r f(s)\,s^{m-1}ds.$$

Now suppose, in order to derive a contradiction, that (10.3) holds. Then we have

$$\begin{aligned} \left\{\int_0^\infty\left(\int_0^r f(s)\,s^{m-1}ds\right)^q \ r^{(\alpha-m)q-\gamma q} \ r^{n-1}dr\right\}^{\frac{1}{q}} &\lesssim \left\{\int_{\mathbb{R}^m} I_\alpha^m f(x)^q \ |x|^{-\gamma q} \ dx\right\}^{\frac{1}{q}} \\ \leq N_{p,q}(w,v)\left\{\int_{\mathbb{R}^m} f(x)^p \ |x|^{\delta p} \ dx\right\}^{\frac{1}{p}} &= N_{p,q}(w,v)\left\{\int_0^\infty f(s) \ s^{\delta p+m-1}ds\right\}^{\frac{1}{p}} \end{aligned}$$

for all $f\geq 0$. With $g(s)\equiv f(s)\,s^{m-1}$, this last inequality can be rewritten as

$$\left\{\int_0^\infty\left(\int_0^r g(s)\,ds\right)^q \ v(r)\,dr\right\}^{\frac{1}{q}}\lesssim N_{p,q}(w,v)\left\{\int_0^\infty g(s) \ u(s)\,ds\right\}^{\frac{1}{p}} \tag{10.10}$$

for all $g \geq 0$, and where the weights are given by $v(r) = r^{(\alpha-m)q-\gamma q+n-1}$ and $u(s) = s^{\delta p}$. By a result of Maz'ja [Maz], the two weight Hardy inequality (10.10) with $q < p$ holds if and only if

$$\int_0^\infty \left[\left(\int_0^r u^{1-p'} \right)^{\frac{1}{p'}} \left(\int_r^\infty v \right)^{\frac{1}{p}} \right]^\rho v(r)\, dr < \infty,$$

where $\frac{1}{\rho} = \frac{1}{q} - \frac{1}{p} > 0$. But since the weights u and v are power functions, the integrand above is also a power function, and hence cannot belong to any Lebesgue space $L^\rho(0,\infty)$, thus providing the required contradiction. □

10.3. **Optimal powers of Muckenhoupt characteristics.** Recall the one parameter two-tailed characteristic $\widehat{A}_{p,q}$ given above by

$$\widehat{A}_{p,q}(v,w) \equiv \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} [\widehat{s}_Q(x)\, w(x)]^q \ dx \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} \left[\widehat{s}_Q(x)\, v(x)^{-1}\right]^{p'} \ dx \right)^{\frac{1}{p'}},$$

where

$$\widehat{s}_Q(x) \equiv \left(1 + \frac{|x-c_Q|}{|Q|^{\frac{1}{n}}} \right)^{\alpha-n}, \qquad c_Q \text{ is the center of } Q,$$

and

$$\begin{aligned} \frac{1}{q} &= \frac{1}{p} - \frac{\alpha}{n}, \\ \text{i.e. } \frac{1}{q} + \frac{1}{p'} &= 1 - \frac{\alpha}{n}, \end{aligned}$$

From Theorem 2 we know that the characteristic $\widehat{A}_{p,q}(w,v)$ is finite *if and only if* the following norm inequality for the fractional integral I_α^n holds:

$$\left\{ \int_{\mathbb{R}^n} I_\alpha^n f(x)^q \ w(x)^q \ dx \right\}^{\frac{1}{q}} \leq C_{p,q}(v,w) \left\{ \int_{\mathbb{R}^n} f(x)^p \ v(x)^p \ dx \right\}^{\frac{1}{p}}. \tag{10.11}$$

Moreover, it is claimed there that

$$C_{p,q}(v,w) \approx \widehat{A}_{p,q}(v,w), \tag{10.12}$$

and this equivalence can be verified by carefully tracking the constants in the proof of Theorem 1 in [SaWh]. We also have the same consequence for the one-tailed characteristic.

Porism: The same arguments as used in the proof of Theorem 1 in [SaWh] also prove that

$$C_{\alpha,\beta}(v,w) \approx \overline{A}_{p,q}(v,w),$$

where $\overline{A}_{p,q}$ is the one-tailed characteristic,

$$\begin{aligned} \overline{A}_{p,q}(v,w) &= \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} \widehat{s}_Q^{p'} v^{-p'} \right)^{\frac{1}{p'}} \\ &\quad + \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} \widehat{s}_Q^{q} w^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q v^{-p'} \right)^{\frac{1}{p'}}. \end{aligned}$$

10.3.1. *Comparison with the inequality of Lacey, Moen, Pérez and Torres.* Here we give a simple and instructive proof of the 'A_1 conjecture' in the setting of fractional integrals. Recall that from [LaMoPeTo] we have the estimate

$$C_{p,q}(w) \lesssim A_{p,q}(w)^{1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}}, \tag{10.13}$$

and if we restrict the two weight result above to the case $w = v$, we have the equivalence

$$C_{\alpha,\beta}(w,w) \approx \widehat{A}_{p,q}(w,w).$$

Thus the estimate (10.13) is equivalent to

$$\widehat{A}_{p,q}(w) \lesssim A_{p,q}(w)^\rho, \tag{10.14}$$

and also equivalent to

$$\overline{A}_{p,q}(w) \lesssim A_{p,q}(w)^{\rho}, \tag{10.15}$$

where $\rho \equiv 1 + \max\left\{\frac{p'}{q}, \frac{q}{p'}\right\}$. Written out in full, one half of inequality (10.15) is

$$\begin{aligned} &\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}\widehat{s}_Q^q w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{1}{p'}} \\ \lesssim\ &\left\{\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{1}{p'}}\right\}^{\rho}. \end{aligned} \tag{10.16}$$

Claim 2. *The inequality (10.15) holds directly, without any reference to norm inequalities at all.*

Proof. Take the q^{th} power of the left hand side of (10.16) and fix a cube Q which comes close to achieving the supremum over all cubes. Then we write

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}\widehat{s}_Q^q w^q\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{q}{p'}} \\ \lesssim\ &\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{q}{p'}}\sum_{k=0}^{\infty}2^{k[(\alpha-n)q+n]}\frac{1}{|2^kQ|}\int_{2^kQ}w^q \\ \lesssim\ &\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{q}{p'}}\sum_{k=0}^{\infty}2^{k[(\alpha-n)q+n]}A_{p,q}(w)^q\left(\frac{1}{|2^kQ|}\int_{2^kQ}w^{-p'}\right)^{-\frac{q}{p'}} \\ =\ &A_{p,q}(w)^q\sum_{k=0}^{\infty}\left(\frac{|Q|_{w^{-p'}}}{|2^kQ|_{w^{-p'}}}\right)^{\frac{q}{p'}} \lesssim A_{p,q}(w)^q\sum_{k=0}^{\infty}\left(2^{-k\delta}\right)^{\frac{q}{p'}} \\ \lesssim\ &A_{p,q}(w)^q\frac{1}{1-2^{-\delta\frac{q}{p'}}} \approx A_{p,q}(w)^q\frac{1}{\delta}, \end{aligned} \tag{10.17}$$

where δ is the reverse doubling exponent for the A_p weight $w^{-p'}$,

$$|Q|_{w^{-p'}} \leq C2^{-k\delta}\left|2^kQ\right|_{w^{-p'}} \text{ for all cubes } Q.$$

Now we claim that the reverse doubling exponents $\delta = \delta\left(w^{-p'}\right)$ and $\delta(w^q)$ for the weights $w^{-p'}$ and w^q satisfy

$$\frac{1}{\delta\left(w^{-p'}\right)} \leq C_{p,q,n}\ A_{p,q}(w)^{p'} \text{ and } \frac{1}{\delta(w^q)} \leq C_{p,q,n}\ A_{p,q}(w)^q. \tag{10.18}$$

Indeed, we have for all $f \geq 0$ and any $0 < \varepsilon < 1$,

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q f\right)\left(\int_Q w^q\right)^{\frac{1}{q}} \\ =\ &\left(\frac{1}{|Q|}\int_Q (f^{\varepsilon}w)\left(f^{1-\varepsilon}\right)\left(w^{-1}\right)\right)\left(\int_Q w^q\right)^{\frac{1}{q}} \\ \leq\ &\left(\frac{1}{|Q|}\int_Q (f^{\varepsilon}w)^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q \left(f^{1-\varepsilon}\right)^{\frac{pq}{q-p}}\right)^{\frac{1}{p}-\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{1}{p'}}\left(\int_Q w^q\right)^{\frac{1}{q}} \\ =\ &\left(\frac{1}{|Q|}\int_Q w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{1}{p'}}\left(\int_Q f^{\varepsilon q}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q f^{(1-\varepsilon)\frac{pq}{q-p}}\right)^{\frac{1}{p}-\frac{1}{q}} \\ \leq\ &A_{p,q}(w)\left(\int_Q f^{\varepsilon q}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q f^{(1-\varepsilon)\frac{pq}{q-p}}\right)^{\frac{1}{p}-\frac{1}{q}}, \end{aligned}$$

and now plugging in $f=\mathbf{1}_{\frac{1}{3}Q}$ we get

$$\begin{aligned}\left(\frac{\left|\frac{1}{3}Q\right|}{|Q|}\right)\left(\int_Q w^q\right)^{\frac{1}{q}} &\leq A_{p,q}(w)\left(\int_{\frac{1}{3}Q} w^q\right)^{\frac{1}{q}}\left(\frac{\left|\frac{1}{3}Q\right|}{|Q|}\right)^{\frac{1}{p}-\frac{1}{q}};\\ \frac{|Q|_{w^q}}{\left|\frac{1}{3}Q\right|_{w^q}} &\leq 3^{n\left(1+\frac{q}{p'}\right)}A_{p,q}(w)^q .\end{aligned}$$

With $\ell(Q)$ denoting the side length of Q, we have

$$\begin{aligned}|3Q|_{w^q} &\leq \sum_{\alpha\in\{-1,0,1\}^n\setminus(0,\dots,0)}|3(Q+\ell(Q)\mathbf{e}_1)|_{w^q}\\ &\leq \sum_{\alpha\in\{-1,0,1\}^n\setminus(0,\dots,0)}3^{n\left(1+\frac{q}{p'}\right)}A_{p,q}(w)^q|Q+\ell(Q)\mathbf{e}_1|_{w^q}\\ &= 3^{n\left(1+\frac{q}{p'}\right)}A_{p,q}(w)^q|3Q\setminus Q|_{w^q} ,\end{aligned}$$

and so

$$\frac{|Q|_{w^q}}{|3Q|_{w^q}}=\frac{|3Q|_{w^q}-|3Q\setminus Q|_{w^q}}{|3Q|_{w^q}}\leq 1-\frac{1}{3^{n\left(1+\frac{q}{p'}\right)}A_{p,q}(w)^q}\equiv\gamma\in\left(1-3^{-n\left(1+\frac{q}{p'}\right)},1\right).$$

Iterating, we get

$$\frac{\left|\frac{1}{3^k}Q\right|_{w^q}}{|Q|_{w^q}}=\frac{\left|\frac{1}{3^k}Q\right|_{w^q}}{\left|\frac{1}{3^{k-1}}Q\right|_{w^q}}\frac{\left|\frac{1}{3^{k-1}}Q\right|_{w^q}}{\left|\frac{1}{3^{k-2}}Q\right|_{w^q}}\dots\frac{\left|\frac{1}{3}Q\right|_{w^q}}{|Q|_{w^q}}\leq\gamma^k,$$

from which we obtain

$$\frac{\left|\frac{1}{2^k}Q\right|_{w^q}}{|Q|_{w^q}}=\frac{\left|\frac{1}{3^{\frac{\ln 2}{\ln 3}k}}Q\right|_{w^q}}{|Q|_{w^q}}\leq C\gamma^{\frac{\ln 2}{\ln 3}k}=C2^{\frac{\ln\gamma}{\ln 3}k}=C2^{-k\delta},$$

where

$$\delta=\delta(w^q)=\frac{\ln\frac{1}{\gamma}}{\ln 3}=\frac{1}{\ln 3}\ln\frac{1}{1-\frac{1}{3^{n\left(1+\frac{q}{p'}\right)}A_{p,q}(w)^q}}\approx\frac{1}{A_{p,q}(w)^q}.$$

This proves the second assertion in (10.18), and the proof of the first assertion is similar.

Thus from (10.17) we obtain

$$\begin{aligned}\overline{A}_{p,q}(w) &= \sup_Q\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}\widehat{s}_Q^q w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{1}{p'}}\\ &\leq C_{p,q,n}\ A_{p,q}(w)\left(\frac{1}{\delta(w^{-p'})}\right)^{\frac{1}{q}}\\ &\leq C_{p,q,n}\ A_{p,q}(w)^{1+\frac{q}{p'}} .\end{aligned}$$

Similarly we obtain that the expression

$$\begin{aligned}&\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}w^q\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q\left[\widehat{s}_Q w^{-1}\right]^{p'}\right)^{\frac{1}{p'}}\\ =&\sup_{Q\subset\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q\left[\widehat{s}_Q w^{-1}\right]^{p'}\right)^{\frac{1}{p'}}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}\left[w^{-1}\right]^{-(q')'}\right)^{\frac{1}{(q')'}}\end{aligned}$$

is dominated by

$$C_{q',p',n}\ A_{q',p'}\left(w^{-1}\right)^{1+\frac{p'}{q}}=C_{p,q,n}\ A_{p,q}(w)^{1+\frac{p'}{q}} .$$

This completes the proof of the claim. □

Thus we have given in the Claim above a simple direct proof of the following theorem using the proof of Theorem 1 in [SaWh].

Theorem 16. *(Lacey, Moen, Pérez and Torres* [LaMoPeTo]*) With p,q,n as above we have*

$$C_{p,q}(w) \leq C_{p,q,n}\ A_{p,q}(w)^{1+\max\left\{\frac{p'}{q},\frac{q}{p'}\right\}}.$$

10.4. **The product fractional integral, the Dirac mass and a modified example.** Here we first consider both the two weight norm inequality (3.3) and the two-tailed characteristic (3.5) in the special case when $\sigma = \delta_{(0,0)}$, and show that they are equivalent in this case. Then we revisit Example to show that we can arrange for even the two-tailed characteristic to be finite when the operator norm is infinite. When $\sigma = \delta_{(0,0)}$ the two weight norm inequality (3.3) yields

$$\begin{aligned} \mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right) &= \left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} I_{\alpha,\beta}^{m,n}\delta_{(0,0)}(x,y)^q\ d\omega(x,y)\right\}^{\frac{1}{q}} \\ &= \left\{\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}\left(|x|^{\alpha-m}|y|^{\beta-n}\right)^q\ d\omega(x,y)\right\}^{\frac{1}{q}}, \end{aligned}$$

and since $\widehat{s}_{I\times J}(x,y) \equiv \left(1+\frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)^{\alpha-m}\left(1+\frac{|y-c_J|}{|J|^{\frac{1}{n}}}\right)^{\beta-n}$, (3.5) yields

$$\begin{aligned} &\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right) \\ &= \sup_{I\times J\subset\mathbb{R}^m\times\mathbb{R}^n}|I|^{\frac{\alpha}{m}-1}|J|^{\frac{\beta}{n}-1}\left(\iint_{\mathbb{R}^m\times\mathbb{R}^n}\widehat{s}_{I\times J}(x,y)^q\,d\omega(x,y)\right)^{\frac{1}{q}}\left(\iint_{\mathbb{R}^m\times\mathbb{R}^n}\widehat{s}_{I\times J}(u,t)^{p'}\,d\delta_{(0,0)}(u,t)\right)^{\frac{1}{p'}} \\ &= \sup_{I\times J\subset\mathbb{R}^m\times\mathbb{R}^n}\left(\iint_{\mathbb{R}^m\times\mathbb{R}^n}|I|^{\left(\frac{\alpha}{m}-1\right)q}\left(1+\frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)^{(\alpha-m)q}|J|^{\left(\frac{\beta}{n}-1\right)q}\left(1+\frac{|y-c_J|}{|J|^{\frac{1}{n}}}\right)^{(\beta-n)q}d\omega(x,y)\right)^{\frac{1}{q}} \\ &\qquad\times\left(1+\frac{|c_I|}{|I|^{\frac{1}{m}}}\right)^{\alpha-m}\left(1+\frac{|c_J|}{|J|^{\frac{1}{n}}}\right)^{\beta-n} \\ &= \sup_{I\times J\subset\mathbb{R}^m\times\mathbb{R}^n}\left(\iint_{\mathbb{R}^m\times\mathbb{R}^n}\left(|I|^{\frac{1}{m}}+|x-c_I|\right)^{(\alpha-m)q}\left(1+\frac{|c_I|}{|I|^{\frac{1}{m}}}\right)^{(\alpha-m)q}\left(|J|^{\frac{1}{n}}+|y-c_J|\right)^{(\beta-n)q}\left(1+\frac{|c_J|}{|J|^{\frac{1}{n}}}\right)^{(\beta-n)q}d\omega(x,y)\right. \end{aligned}$$

If $I\times J$ has center $(c_I,c_J)=(0,0)$, then this supremum is equal to or greater than

$$\left(\iint_{\mathbb{R}^m\times\mathbb{R}^n}|x|^{(\alpha-m)q}|y|^{(\beta-n)q}\,d\omega(x,y)\right)^{\frac{1}{q}} = \mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right).$$

Thus from Lemma 3 we conclude that

$$\mathbb{N}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right) \approx \widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}\left(\delta_{(0,0)},\omega\right).$$

Now we refine Example 1 given in the introduction.

Example 2. *Let $0<\alpha,\beta<1$ and $1<p,q<\infty$. Given $0<\rho<\infty$, define a weight pair $(\sigma_\rho,\omega_\rho)$ in the plane $\mathbb{R}^2$ by*

$$\sigma_\rho \equiv \sum_{P\in\mathcal{P}_\rho}\delta_{-P}\ \text{and}\ \omega_\rho \equiv \sum_{P\in\mathcal{P}_\rho}\delta_P \qquad \text{where } \mathcal{P}_\rho = \left\{\left(2^k,2^{-\rho k}\right)\right\}_{k=1}^\infty.$$

If $R=I\times J$ is a rectangle in the plane $\mathbb{R}\times\mathbb{R}$ with sides parallel to the axes that is symmetric about the origin $(0,0)$ and satisfies $R\cap\mathcal{P}_{\rho}=\left\{\left(2^{N},2^{-\rho N}\right)\right\}$ for some $N\geq 1$, then it follows that

$$\begin{aligned}
&\left|I\right|^{\alpha-1}\left|J\right|^{\beta-1}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{p^{\prime}}d\sigma_{\rho,\mu}\right)^{\frac{1}{p^{\prime}}}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{q}d\omega_{\rho,\mu}\right)^{\frac{1}{q}}\\
\lesssim\ &\left(2^{N}\right)^{\alpha-1}\left(2^{-\rho N}\right)^{\beta-1}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left[\left(1+\frac{\left|x\right|}{2^{N}}\right)^{\alpha-1}\left(1+\frac{\left|y\right|}{2^{-\rho N}}\right)^{\beta-1}\right]^{p^{\prime}}d\sigma_{\rho}\left(x,y\right)\right)^{\frac{1}{p^{\prime}}}\\
&\times\left(\iint_{\mathbb{R}\times\mathbb{R}}\left[\left(1+\frac{\left|x\right|}{2^{N}}\right)^{\alpha-1}\left(1+\frac{\left|y\right|}{2^{-\rho N}}\right)^{\beta-1}\right]^{q}d\omega_{\rho}\left(x,y\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Now we compute that

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}}\left[\left(1+\frac{\left|x\right|}{2^{N}}\right)^{\alpha-1}\left(1+\frac{\left|y\right|}{2^{-\rho N}}\right)^{\beta-1}\right]^{q}d\omega_{\rho}\left(x,y\right)\\
=\ &\sum_{k=1}^{\infty}\left[\left(1+\frac{k^{\mu}2^{k}}{2^{N}}\right)^{\alpha-1}\left(1+\frac{2^{-\rho k}}{2^{-\rho N}}\right)^{\beta-1}\right]^{q}\\
\approx\ &\sum_{k=1}^{N}\left[\left(2^{\rho(N-k)}\right)^{\beta-1}\right]^{q}+\sum_{k=N}^{\infty}\left[\left(2^{k-N}\right)^{\alpha-1}\right]^{q}=C,
\end{aligned}$$

and so

$$\left|I\right|^{\alpha-1}\left|J\right|^{\beta-1}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{p^{\prime}}d\sigma_{\rho,\mu}\right)^{\frac{1}{p^{\prime}}}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{q}d\omega_{\rho,\mu}\right)^{\frac{1}{q}}\lesssim\left(2^{N}\right)^{\alpha-1}\left(2^{-\rho N}\right)^{\beta-1}$$

is uniformly bounded in N provided $\rho\leq\frac{1-\alpha}{1-\beta}$. More generally, suppose now that $R=I\times J$ is a rectangle in the plane $\mathbb{R}\times\mathbb{R}$ with sides parallel to the axes that is symmetric about the origin $(0,0)$ and satisfies $R\cap\mathcal{P}_{\rho}=\left\{\left(2^{k},2^{-\rho k}\right)\right\}_{k=L+1}^{L+N}$ for some $L\geq 0$ and $N\geq 1$. We compute

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{q}d\omega_{\rho,\mu}=\iint_{\mathbb{R}\times\mathbb{R}}\left[\left(1+\frac{\left|x\right|}{2^{L+N}}\right)^{\alpha-1}\left(1+\frac{\left|y\right|}{2^{-\rho(L+1)}}\right)^{\beta-1}\right]^{q}d\omega_{\rho,\mu}\left(x,y\right)\\
=\ &\sum_{k=1}^{\infty}\left[\left(1+\frac{2^{k}}{2^{L+N}}\right)^{\alpha-1}\left(1+\frac{2^{-\rho k}}{2^{-\rho(L+1)}}\right)^{\beta-1}\right]^{q}\\
\approx\ &\sum_{k=1}^{L}\left[\left(2^{\rho(L-k)}\right)^{\beta-1}\right]^{q}+\sum_{k=L+1}^{L+N}1^{q}+\sum_{k=L+N+1}^{\infty}\left[\left(2^{k-L-N}\right)^{\alpha-1}\right]^{q}\approx N,
\end{aligned}$$

and then it follows that

$$\begin{aligned}
&\left|I\right|^{\alpha-1}\left|J\right|^{\beta-1}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{p^{\prime}}d\sigma_{\rho,\mu}\right)^{\frac{1}{p^{\prime}}}\left(\iint_{\mathbb{R}\times\mathbb{R}}\left(\widehat{s}_{I\times J}\right)^{q}d\omega_{\rho,\mu}\right)^{\frac{1}{q}}\\
\lesssim\ &\left(2^{L+N}\right)^{\alpha-1}\left(2^{-\rho(L+1)}\right)^{\beta-1}N^{\frac{1}{q}}N^{\frac{1}{p^{\prime}}}.
\end{aligned}$$

This expression is uniformly bounded, and hence $\widehat{\mathbb{A}}_{p,q}^{(\alpha,1),(\beta,1)}(\sigma_\rho,\omega_\rho)$ *is bounded, provided* $\rho \leq \frac{1-\alpha}{1-\beta}$. *On the other hand, the function* $f(x,y) \equiv c\left(\frac{1}{\log_2|x|}\right)$ *satisfies*

$$\iint_{\mathbb{R}\times\mathbb{R}} f^p d\sigma_\rho = c\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^p = 1,$$

hence $f \in L^p(\sigma_\rho)$ *with norm* 1 *provided* $p>1$ *and* $c>0$ *is chosen appropriately, while the strong dyadic fractional maximal function satisfies*

$$\begin{aligned} \mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}(f\sigma_\rho)\left(2^N,2^{-\rho N}\right) &\geq \left|\left(-2^N,2^N\right)\right|^{\alpha-1}\left|\left(-2^{-\rho N},2^{-\rho N}\right)\right|^{\beta-1}\int_{[-2^N,2^N)\times[-2^{-\rho N},2^{-\rho N})} f d\sigma_\rho \\ &= 2^{N(\alpha-1-\rho(\beta-1))}(2)^{\alpha-1}\sum_{k=1}^{N}\frac{1}{k} \\ &\geq c2^{N(\alpha-1-\rho(\beta-1))}(1+\ln N) \geq c>0, \end{aligned}$$

for all $N \geq 1$ *provided*

$$\rho \geq \frac{1-\alpha}{1-\beta}.$$

Thus if $\rho = \frac{1-\alpha}{1-\beta}$, *even the* weak type *operator norm of the dyadic fractional maximal operator*$\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}$ *is infinite since* $\mathcal{M}_{\alpha,\beta}^{\mathrm{dy}}(f\sigma_\rho) > c > 0$ *on the set* $\mathcal{P}_\rho$ *of infinite* ω_ρ*-measure.*

10.5. **Reverse doubling and Muckenhoupt type conditions.** Recall the reverse doubling condition with exponent $\varepsilon>0$, and rectangle reverse doubling condition with exponent pair $\left(\varepsilon^1,\varepsilon^2\right)$.

Definition 3. *We say that a measure* μ *on* $\mathbb{R}^m$ *satisfies the* reverse doubling condition $(R_{ev}D_{oub})$ *with exponent* $\varepsilon>0$ *if*

$$|sI|_\mu \leq Cs^{m\varepsilon}\ \ |I|_\mu\,, \qquad 0<s<1,$$

for some constant C*; and we say that a measure* μ *on* $\mathbb{R}^m\times\mathbb{R}^n$ *satisfies the* rectangle reverse doubling condition $(R_{ect}R_{ev}D_{oub})$ *with exponent pair* $\left(\varepsilon^1,\varepsilon^2\right)$ *if*

$$|sI\times tJ|_\mu \leq Cs^{m\varepsilon^1}t^{n\varepsilon^2}\ \ |I\times J|_\mu\,, \qquad 0<s,t<1,$$

for some constant C. *If* $\varepsilon=\varepsilon^1=\varepsilon^2$, *then we say simply that* ε *is the reverse doubling exponent for* μ.

Note that unlike doubling measures, nontrivial reverse doubling measures can vanish on open subsets. In particular, Cantor measures are typically reverse doubling, while never doubling. If both weights are reverse doubling, then the two-tailed, one-tailed, and no-tailed Muckenhoupt conditions are all equivalent.

Lemma 5. *Suppose that* σ *and* ω *are reverse rectangle doubling measures on* $\mathbb{R}^n$, *and that*

$$0<\alpha<m, 0<\beta<n, 1<p,q<\infty.$$

Then we have the following equivalence:

$$\widehat{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega) \approx \overline{\mathbb{A}}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega) \approx \mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma,\omega)\,.$$

Proof. We first consider the one-parameter equivalence

$$\widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega) \approx \overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega) \approx \mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega)\,, \tag{10.19}$$

since the extension to two parameters will follow the same proof with obvious modifications. Since $\mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega) \leq \overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega) \leq \widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)$, it suffices to prove

$$\widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega) \lesssim \overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega) \lesssim \mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega)\,, \tag{10.20}$$

and we begin with the second inequality $\overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega) \lesssim \mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega)$ in (10.20).

So fix a cube I which comes close to achieving the supremum in $\overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)$ over all cubes, and suppose the tail $\widehat{s}_I$ occurs in the factor for ω. We have

$$\begin{aligned}
&\sum_{k=0}^{\infty}2^{k[(\alpha-m)q+m]}\frac{1}{|2^kI|}\int_{2^kI}d\omega(x)\\
=&\sum_{k=0}^{\infty}2^{k[(\alpha-m)q+m]}\frac{1}{|2^kI|}\sum_{\ell=0}^{k}\int_{2^\ell I\setminus 2^{\ell-1}I}d\omega(x)=\sum_{\ell=0}^{\infty}\left(\sum_{k=\ell}^{\infty}2^{k(\alpha-m)q}\right)\int_{2^\ell I\setminus 2^{\ell-1}I}d\omega(x)\\
\approx&\sum_{\ell=0}^{\infty}2^{\ell(\alpha-m)q}\int_{2^\ell I\setminus 2^{\ell-1}I}d\omega(x)\approx\frac{1}{|I|}\int_{\mathbb{R}^m}\left(1+\frac{|x-c_I|}{|I|^{\frac{1}{m}}}\right)^{(\alpha-m)q}d\omega(x)=\frac{1}{|I|}\int_{\mathbb{R}^m}\widehat{s}_I^{\,q}d\omega.
\end{aligned}$$

since $\alpha<m$. Then we write

(10.21)

$$\begin{aligned}
&\overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^q\approx\left\{|I|^{\frac{\alpha}{m}-\Gamma}\left(\frac{1}{|I|}\int_{\mathbb{R}^m}\widehat{s}_I^{\,q}d\omega\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int_I d\sigma\right)^{\frac{1}{p'}}\right\}^q\\
\approx&\ |I|^{\left(\frac{\alpha}{m}-\Gamma\right)q}\sum_{k=0}^{\infty}2^{k[(\alpha-m)q+m]}\left(\frac{1}{|2^kI|}\int_{2^kI}d\omega\right)\left(\frac{1}{|I|}\int_I d\sigma\right)^{\frac{q}{p'}}\\
\lesssim&\ |I|^{\left(\frac{\alpha}{m}-\Gamma\right)q}\sum_{k=0}^{\infty}2^{k[(\alpha-m)q+m]}\left(\frac{1}{|2^kI|}\int_{2^kI}d\omega\right)\left(\frac{1}{|I|}C2^{-k\delta}\left|2^kI\right|_\sigma\right)^{\frac{q}{p'}}\\
=&\ \sum_{k=0}^{\infty}2^{-k(\alpha-m\Gamma)q}\left|2^kI\right|^{\left(\frac{\alpha}{m}-\Gamma\right)q}2^{k[(\alpha-m)q+m]}\left(\frac{1}{|2^kI|}\int_{2^kI}d\omega\right)2^{km\frac{q}{p'}}\left(\frac{1}{|2^kI|}C2^{-k\delta}\left|2^kI\right|_\sigma\right)^{\frac{q}{p'}}\\
=&\ \sum_{k=0}^{\infty}C2^{-km\varepsilon}2^{-k(\alpha-m\Gamma)q}2^{k[(\alpha-m)q+m]}2^{km\frac{q}{p'}}\left\{\left|2^kI\right|^{\left(\frac{\alpha}{m}-\Gamma\right)q}\left(\frac{1}{|2^kI|}\int_{2^kI}d\omega\right)\left(\frac{1}{|2^kI|}\left|2^kI\right|_\sigma\right)^{\frac{q}{p'}}\right\},
\end{aligned}$$

where $\varepsilon>0$ is the reverse doubling exponent for the weight σ,

$$|Q|_\sigma\leq C2^{-km\varepsilon}\left|2^kQ\right|_\sigma\text{ for all cubes }Q.$$

We then dominate the term in braces by $\mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega)$ to obtain that

$$\overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^q\lesssim\mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega)^q\sum_{k=0}^{\infty}2^{-km\varepsilon}\lesssim\mathbb{A}_{p,q}^{\alpha,m}(\sigma,\omega)^q,$$

since $-k(\alpha-m\Gamma)q+k[(\alpha-m)q+m]+km\frac{q}{p'}=0$, and this completes the proof of the second inequality in (10.20). Note that if the cube I which comes close to achieving the supremum in $\overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)$ over all cubes has the tail $\widehat{s}_I$ occurrings in the factor for σ, then the above argument applies using reverse doubling for the weight ω.

Now we consider the first inequality in (10.20). Here we can use reverse doubling for either σ or ω, and we will use a decomposition that uses reverse doubling for ω with exponent $\varepsilon>0$, i.e. $|Q|_\omega\leq C2^{-km\varepsilon}\left|2^kQ\right|_\omega$ for all cubes Q. This then has the analogous consequence for the ω-integrals with tails:

$$\begin{aligned}
\int_{\mathbb{R}^m}\widehat{s}_I^{\,q}d\omega&\approx\sum_{k=0}^{\infty}2^{k[(\alpha-m)q]}\left(\int_{2^kI}d\omega\right)\leq\sum_{k=0}^{\infty}2^{k[(\alpha-m)q]}\left(C2^{-\ell m\varepsilon}\int_{2^{k+\ell}I}d\omega\right)\\
&=C2^{-\ell m\varepsilon}\sum_{k=0}^{\infty}2^{k[(\alpha-m)q]}\left(\int_{2^{k+\ell}I}d\omega\right)\approx C2^{-\ell m\varepsilon}\int_{\mathbb{R}^m}\widehat{s}_{2^\ell I}^{\,q}d\omega.
\end{aligned}$$

Now let I be a cube which comes close to achieving the supremum in $\widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)$ over all cubes. Then we have

$$\begin{aligned}
&\widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'} \approx \left\{ |I|^{\frac{\alpha}{m}-\Gamma} \left(\frac{1}{|I|} \int_{\mathbb{R}^m} \widehat{s}_I^q d\omega \right)^{\frac{1}{q}} \left(\frac{1}{|I|} \int_{\mathbb{R}^m} \widehat{s}_I^{p'} d\sigma \right)^{\frac{1}{p'}} \right\}^{p'} \\
\approx\ & |I|^{\left(\frac{\alpha}{m}-\Gamma\right)p'} \left(\frac{1}{|I|} \int_{\mathbb{R}^m} \widehat{s}_I^q d\omega \right)^{\frac{p'}{q}} \sum_{\ell=0}^{\infty} 2^{\ell\left[(\alpha-m)p'+m\right]} \left(\frac{1}{|2^\ell I|} \int_{2^\ell I} d\sigma \right) \\
\lesssim\ & |I|^{\left(\frac{\alpha}{m}-\Gamma\right)p'} \sum_{\ell=0}^{\infty} C 2^{-\ell m\varepsilon} 2^{\ell\left[(\alpha-m)p'+m\right]} \left(\frac{1}{|I|} \int_{\mathbb{R}^m} \widehat{s}_{2^\ell I}^q d\omega \right)^{\frac{p'}{q}} \left(\frac{1}{|2^\ell I|} \int_{2^\ell I} d\sigma \right) \\
=\ & \sum_{\ell=0}^{\infty} C 2^{-\ell m\varepsilon} 2^{-\ell(\alpha-m\Gamma)p'} 2^{\ell\left[(\alpha-m)p'+m\right]} 2^{\ell m \frac{p'}{q}} \left\{ \left|2^\ell I\right|^{\left(\frac{\alpha}{m}-\Gamma\right)p'} \left(\frac{1}{|2^\ell I|} \int_{\mathbb{R}^m} \widehat{s}_{2^\ell I}^q d\omega \right)^{\frac{p'}{q}} \left(\frac{1}{|2^\ell I|} \int_{2^\ell I} d\sigma \right) \right\}.
\end{aligned}$$

We then dominate the term in braces by $\overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'}$ to obtain that

$$\widehat{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'} \lesssim \overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'} \sum_{\ell=0}^{\infty} 2^{-\ell m\varepsilon} \lesssim \overline{\mathbb{A}}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'},$$

since $-\ell(\alpha-m\Gamma)p' + \ell[(\alpha-m)p'+m] + \ell m\frac{p'}{q} = 0$, and this completes the proof of the first inequality in (10.20), which in turn completes the proof of the one parameter equivalence (10.19).

In order to prove the two parameter equivalence in Lemma 5, we begin with

$$\sum_{k,j=0}^{\infty} 2^{k[(\alpha-m)q+m]+j[(\beta-n)q+n]} \frac{1}{|2^k I \times 2^j J|} \int_{2^k I\times 2^j J} d\omega(x) \approx \frac{1}{|I\times J|} \int_{\mathbb{R}^m\times\mathbb{R}^n} \widehat{s}_{I,J}^q d\omega,$$

and proceed with the two parameter analogue of (10.21), followed by the straightforward modifications of the remaining arguments. This completes our proof of Lemma 5. □

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