Weighted Inequality of Product Fractional Integrals for Radial Functions

by Zipeng Wang

Abstract

We prove that the weighted norm regularity of the strong fractional integral operator, whose kernel has singularity appeared on every coordinate subspace, is implied by the associated Mouckenhoupt characteristic, for radially symmetric functions.

1 Introduction and Statement of Main Result

In the present paper, we study the regularity of so-called *strong fractional integral operator*, satisfying certain characteristics on the product space

$$\mathbb{R}^{\mathbf{N}} \;=\; \mathbb{R}^{\mathbf{N}_1} \times \mathbb{R}^{\mathbf{N}_2} \times \cdots \times \mathbb{R}^{\mathbf{N}_n}. \tag{1. 1}$$

Let

$$0 < \alpha_i < \mathbf{N}_i, \qquad i = 1, 2, \ldots, n \qquad \text{and} \qquad \alpha \;=\; \alpha_1 + \alpha_2 + \cdots + \alpha_n. \tag{1. 2}$$

The strong fractional integral operator $\mathbf{I}_\alpha$ is defined by

$$\Big(\mathbf{I}_\alpha f\Big)(x) \;=\; \int_{\mathbb{R}^{\mathbf{N}}} f(y) \prod_{i=1}^{n} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i} dy \tag{1. 3}$$

whose kernel has singularity appeared on each of the coordinate subspaces. It is clear that

$$\left(\frac{1}{|x|}\right)^{\mathbf{N}-\alpha} \;\lesssim\; \prod_{i=1}^{n} \left(\frac{1}{|x_i|}\right)^{\mathbf{N}_i - \alpha_i}. \tag{1. 4}$$

The weighted norm regularity theory of such *product operators*, commuting with a multi parameters family of dilations, has seen little in progress since the 1980's after a number of pioneering works accomplished by Robert Fefferman and Stein. See [6]-[7] and several references cited there . The area remains largely open for fractional integrals.

We hereby consider

$$\Big\|\omega \mathbf{I}_\alpha f\Big\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \;\lesssim\; \Big\|f\sigma\Big\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \tag{1. 5}$$

for $1 < p \le q < \infty$, where ω^q and $\sigma^{\frac{p}{1-p}}$ are nonnegative, locally integrable functions.

Let $\mathbf{Q}_i$ to be a cube in $\mathbb{R}^{\mathbf{N}_i}$ for every $i = 1, 2, \ldots, n$. We write

$$\mathbf{Q} \;=\; \mathbf{Q}_1 \;\times\; \mathbf{Q}_2 \;\times\; \cdots \;\times\; \mathbf{Q}_n. \tag{1. 6}$$

We say $\omega, \sigma \in \mathbf{A}^{\alpha}_{pqr}$ if

$$\mathbf{A}^{\alpha}_{pqr}(\omega,\sigma) \doteq \sup_{\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}} \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(x)dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)dx\right\}^{\frac{p-1}{pr}} < \infty \quad (1.\ 7)$$

for $1 < p \le q < \infty$ and $r \ge 1$. In particular, we write $\mathbf{A}^{\alpha}_{pq}$ for $\mathbf{A}^{\alpha}_{pq1}$.

Observe that (1. 7) is an n-parameters variant of Muckenhoupt characteristic. For $r > 1$, it is analogue to Fefferman-Phone's condition, initially introduced for $p = q$, on the product space. See Fefferman [4]. On the other hand, for $r = 1$, we shall find that $\omega, \sigma \in \mathbf{A}^{\alpha}_{pq}$ is an necessity for the norm inequality to hold in (1. 5). Moreover, if ω^q and $\sigma^{\frac{p}{1-p}}$ satisfy *product* A_∞-property, then $\mathbf{A}^{\alpha}_{pq}$ is equivalent to $\mathbf{A}^{\alpha}_{pqr}$ for r sufficiently close to 1.

Theorem A: *The norm inequality in (1. 5) implies $\omega, \sigma \in \mathbf{A}^{\alpha}_{pq}$ in (1. 7) for $1 < p \le q < \infty$. Conversely, suppose that $\omega, \sigma \in \mathbf{A}^{\alpha}_{pqr}$ in (1. 7) for $1 < p < q < \infty$ and $r > 1$. The norm inequality holds in (1. 5) for all radially symmetric functions. Moreover, we have*

$$\left\|\omega \mathbf{I}_\alpha f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \lesssim \mathbf{A}^{\alpha}_{pqr}(\omega,\sigma) \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}$$

where the implied constant depends only on p, q, r, α and $\mathbf{N}$.

Sketch of Proof: In section 2, we give some characteristic estimates for $\omega, \sigma \in \mathbf{A}^{\alpha}_{pqr}$ and $r > 1$. In section 3, we introduce a new framework where the frequency space is decomposed into an union of Dyadic cones. The consisting partial sum operator defined on each cone is essentially an one-parameter fractional integral operator. It has an analytic continuation and satisfies the desired regularity. In section 4, we prove a decaying estimate on its norm whenever the function is radially symmetric. The proof is then completed by interpolations on a family of analytic operators.

2 Characteristic Estimates

Let $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$ and $\chi_{\mathbf{Q}}$ to be its characteristic function. Consider

$$f(x) = \chi_{\mathbf{Q}}(x)\sigma(x)^{\frac{1}{1-p}} \in \mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right) \quad (2.\ 1)$$

provided that $\sigma^{\frac{p}{1-p}}$ is locally integrable.

It follows that

$$\begin{aligned} &\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma^{-1}\right)(y) \prod_{i=1}^{n} \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i-\alpha_i} dy \\ &\gtrsim \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-1} \left\{\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(y)dy\right\} \chi_{\mathbf{Q}}(x). \end{aligned} \quad (2.\ 2)$$

Let ω^q to be locally integrable. The norm inequality in (1. 5) together with (2. 2) imply

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-1}\left\{\int_{\mathbf{Q}}\omega^q(x)dx\right\}^{\frac{1}{q}}\left\{\int_{\mathbf{Q}}\left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx\right\}^{\frac{p-1}{p}}\\
=&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\omega^q(x)dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx\right\}^{\frac{p-1}{p}} < \infty.
\end{aligned}\tag{2. 3}$$

Hence that $\omega,\sigma\in\mathbf{A}^\alpha_{pq}$ is necessary. For $r>1$, Hölder inequality implies that $\mathbf{A}^\alpha_{pq}\subset\mathbf{A}^\alpha_{pqr}$. Consider η to be any nonnegative, measurable function satisfying the *product* A_∞-property. Write $\left(x_i,x_i^\dagger\right)\in\mathbb{R}^{\mathbf{N}_i}\times\mathbb{R}^{\mathbf{N}-\mathbf{N}_i}$ for $i=1,2,\ldots,n$. By applying the reverse Hölder inequality introduced in chapter V of the book by Stein [3], on each of the coordinate subspaces, we have

$$\begin{aligned}
\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\eta^r(x)dx\right\}^{\frac{1}{r}} &= \left(\frac{|\mathbf{Q}_1|}{|\mathbf{Q}|}\right)^{\frac{1}{r}}\left\{\int_{\bigotimes_{i=2}^n\mathbf{Q}_i}\left\{\frac{1}{|\mathbf{Q}_1|}\int_{\mathbf{Q}_1}\eta^r\left(x_1,x_1^\dagger\right)dx_1\right\}dx_1^\dagger\right\}^{\frac{1}{r}}\\
&\lesssim \left(\frac{|\mathbf{Q}_1|}{|\mathbf{Q}|}\right)^{\frac{1}{r}}\left\{\int_{\bigotimes_{i=2}^n\mathbf{Q}_i}\left\{\frac{1}{|\mathbf{Q}_1|}\int_{\mathbf{Q}_1}\eta\left(x_1,x_1^\dagger\right)dx_1\right\}^r dx_1^\dagger\right\}^{\frac{1}{r}}\\
&\lesssim \left(\frac{|\mathbf{Q}_1|}{|\mathbf{Q}|}\right)^{\frac{1}{r}}\frac{1}{|\mathbf{Q}_1|}\left\{\int_{\mathbf{Q}_1}\left\{\int_{\bigotimes_{i=2}^n\mathbf{Q}_i}\eta^r\left(x_1,x_1^\dagger\right)dx_1^\dagger\right\}^{\frac{1}{r}}dx_1\right\}\\
&\qquad\qquad\text{by Minkowski integral inequality}\\
&\quad\vdots\\
&\lesssim \frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\eta(x)dx
\end{aligned}\tag{2. 4}$$

for some $r>1$ and every $\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}$. Therefore, if ω^q and $\sigma^{\frac{p}{1-p}}$ satisfy the *product* A_∞-property, $\mathbf{A}^\alpha_{pq}$ is equivalent to $\mathbf{A}^\alpha_{pqr}$ for r sufficiently close to 1.

Let $\omega,\sigma\in\mathbf{A}^\alpha_{pqr}$ in (1. 7) for some $r>1$. Set $r=st$ for $s>1,\ t>1$. We have

$$\begin{aligned}
\left(\mathbf{A}^\alpha_{pqr}(\omega,\sigma)\right)^s &\geq \left\{\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\omega^{qr}(x)dx\right\}^{\frac{1}{qr}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)dx\right\}^{\frac{p-1}{pr}}\right\}^s\\
&= \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i s}{\mathbf{N}_i}-\left(\frac{s}{p}-\frac{s}{q}\right)}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\omega^s\right)^{tq}(x)dx\right\}^{\frac{1}{tq}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{\sigma^s}\right)^{\frac{tp}{p-1}}(x)dx\right\}^{\frac{p-1}{tp}}
\end{aligned}\tag{2. 5}$$

for every $\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}$.

Define

$$\begin{aligned}
\gamma &= \gamma_1+\gamma_2+\cdots+\gamma_n,\\
\frac{\gamma_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right) &= s\left[\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)\right],\qquad i=1,2,\ldots,n.
\end{aligned}\tag{2. 6}$$

Observe that from (2. 5), we have

$$\omega,\sigma \in \mathbf{A}^{\alpha}_{pqr} \implies \omega^s,\sigma^s \in \mathbf{A}^{\gamma}_{pqt}, \qquad r = st, \qquad \mathbf{A}^{\gamma}_{pqt}(\omega^s,\sigma^s) = \left(\mathbf{A}^{\alpha}_{pqr}(\omega,\sigma)\right)^s. \tag{2. 7}$$

On the other hand, we define

$$\beta = \beta_1+\beta_2+\cdots+\beta_n, \qquad \frac{\beta_i}{\mathbf{N}_i} = \frac{1}{p}-\frac{1}{q}, \qquad i=1,2,\ldots,n. \tag{2. 8}$$

Notice that both γ and β in (2. 6) and (2. 8) are well defined, whereas $0<\gamma_i<\mathbf{N}_i$ and $0<\beta_i<\mathbf{N}_i$ for every $i=1,2,\ldots,n$, provided that s is sufficiently close to 1 and $p<q$.

Let $0<\lambda<1$. By using the formulae given respectively in (2. 6) and (2. 8), we have

$$\begin{aligned}
\infty &> \left\{\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\gamma_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} (\omega^s)^{tq}(x)dx\right\}^{\frac{1}{tq}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma^s}\right)^{\frac{tp}{p-1}}(x)dx\right\}^{\frac{p-1}{tp}}\right\}^{\lambda} \\
&= \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\lambda\gamma_i}{\mathbf{N}_i}-\left(\frac{\lambda}{p}-\frac{\lambda}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\omega^{s\lambda}\right)^{\frac{tq}{\lambda}}(x)dx\right\}^{\frac{\lambda}{tq}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma^{s\lambda}}\right)^{\frac{1}{\lambda}\left(\frac{tp}{p-1}\right)}(x)dx\right\}^{\lambda\left(\frac{p-1}{tp}\right)} \\
&\geq \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{(1-\lambda)\beta_i+\lambda\gamma_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\omega^{s\lambda}\right)^{tq}(x)dx\right\}^{\frac{1}{tq}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma^{s\lambda}}\right)^{\frac{tp}{p-1}}(x)dx\right\}^{\frac{p-1}{tp}}
\end{aligned} \tag{2. 9}$$

where the last inequality is obtained by applying Hölder inequality. Hence that

$$\omega,\sigma \in \mathbf{A}^{\alpha}_{pqr} \implies \omega^{s\lambda},\sigma^{s\lambda} \in \mathbf{A}^{(1-\lambda)\beta+\lambda\gamma}_{pqt}, \qquad r=st, \qquad 0<\lambda<1. \tag{2. 10}$$

3 Dilation Estimates and Size of Kernel

Let ξ to be the dual variable of x in the product space for which the inner product

$$x\cdot\xi \doteq x_1\cdot\xi_1 + x_2\cdot\xi_2+\cdots+x_n\cdot\xi_n. \tag{3. 1}$$

Let t_i for $i=1,2,\ldots,n$ to be nonnegative integers. Consider $\varphi\in C_o^\infty(\mathbb{R})$ such that $\varphi(t)\equiv 1$ for $|t|\leq 1$ and $\varphi(t)=0$ for $|t|>2$. We define

$$\delta_{\mathbf{t}}(\xi) = \prod_{i=1}^n \varphi\left(2^{t_i}\frac{|\xi_i|}{|\xi|}\right) - \varphi\left(2^{t_i+1}\frac{|\xi_i|}{|\xi|}\right) \tag{3. 2}$$

which is supported on the Dyadic cone

$$\Lambda_{\mathbf{t}} \doteq \left\{\xi\in\mathbb{R}^{\mathbf{N}} : 2^{-t_i-1}\leq\frac{|\xi_i|}{|\xi|}\leq 2^{-t_i+1},\ i=1,2,\ldots,n\right\}. \tag{3. 3}$$

In particular, we write $\delta_o=\delta_{\mathbf{t}}$ and $\Lambda_o=\Lambda_{\mathbf{t}}$ for $t_1=t_2=\cdots=t_n=0$.

The partial sum operator $\Delta_{\mathbf{t}}$ is defined by its Fourier transform

$$\left(\widehat{\Delta_{\mathbf{t}} f}\right)(\xi) \;=\; \delta_{\mathbf{t}}(\xi)\widehat{f}(\xi). \tag{3. 4}$$

From (1. 2)-(1. 3), we have $\mathbf{I}_\alpha f = f * \Omega^\alpha$ where

$$\Omega^\alpha(x) \;=\; \prod_{i=1}^n \left(\frac{1}{|x_i|}\right)^{\mathbf{N}_i - \alpha_i}. \tag{3. 5}$$

Its Fourier transform equals

$$\prod_{i=1}^n \frac{2^{\frac{\mathbf{N}_i-\alpha_i}{2}}\Gamma\left(\frac{\mathbf{N}_i-\alpha_i}{2}\right)}{2^{\frac{\alpha_i}{2}}\Gamma\left(\frac{\alpha_i}{2}\right)}\left(\frac{1}{|\xi_i|}\right)^{\alpha_i} \;\doteq\; \mathfrak{C}_\alpha \prod_{i=1}^n \left(\frac{1}{|\xi_i|}\right)^{\alpha_i} \tag{3. 6}$$

where Γ in (3. 6) is Gamma function. Recall that $1/\Gamma(s)$ is an entire function with zeros only appeared at $s = 0, -1, -2, \ldots$. We shall therefore consider $\mathfrak{C}_\alpha$ given implicitly in (3. 6) as a generic constant.

For each given $\mathbf{t}$, we have $\Delta_{\mathbf{t}}\mathbf{I}_\alpha f = f * \Omega^\alpha_{\mathbf{t}}$ where

$$\Omega^\alpha_{\mathbf{t}}(x) \;=\; \mathfrak{C}_\alpha \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x\cdot\xi}\delta_{\mathbf{t}}(\xi)\prod_{i=1}^n \left(\frac{1}{|\xi_i|}\right)^{\alpha_i} d\xi. \tag{3. 7}$$

Let $\rho = \rho_1 + \rho_2 + \cdots + \rho_n$ and consider

$$\Omega^{\alpha+\mathbf{i}\rho}_{\mathbf{t}}(x) \;\doteq\; \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x\cdot\xi}\delta_{\mathbf{t}}(\xi)\prod_{i=1}^n \left(\frac{1}{|\xi_i|}\right)^{\alpha_i+\mathbf{i}\rho_i} d\xi. \tag{3. 8}$$

In particular, we write $\Omega^{\alpha+\mathbf{i}\rho}_o = \Omega^{\alpha+\mathbf{i}\rho}_{\mathbf{t}}$ for $t_1 = t_2 = \cdots = t_n = 0$.

Lemma 3.1 *We have*

$$\left|\Omega^{\alpha+\mathbf{i}\rho}_o(x)\right| \;\lesssim\; (1+\rho)^{\mathbf{N}}\left(\frac{1}{|x|}\right)^{\mathbf{N}-\alpha}. \tag{3. 9}$$

Proof: Let $\varphi \in C^\infty_o(\mathbb{R})$ such that $\varphi(t) \equiv 1$ for $|t| \le 1$ and $\varphi(t) = 0$ for $|t| > 2$. We define

$$\phi_j(x) \;=\; \varphi\left(2^{-j}|\xi|\right) - \varphi\left(2^{-j+1}|\xi|\right) \tag{3. 10}$$

for every $j \in \mathbb{Z}$.

From (3. 2), we have $|\xi_i| \approx |\xi|$ for every $i = 1, 2, \ldots, n$ whenever $\xi \in \mathbf{supp}\delta_o = \Lambda_o$ in (3. 3). Recall that $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ in (1. 2). From (3. 8), we have

$$\begin{aligned}\Omega^{\alpha+\mathbf{i}\rho}_o(x) \;&=\; \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x\cdot\xi}\delta_o(\xi)\prod_{i=1}^n \left(\frac{1}{|\xi_i|}\right)^{\alpha_i+\mathbf{i}\rho_i} d\xi \\ &\approx\; \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x\cdot\xi}\delta_o(\xi)\left(\frac{1}{|\xi|}\right)^{\alpha+\mathbf{i}\rho} d\xi \qquad (3.\ 11) \\ &=\; \sum_j \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x\cdot\xi}\delta_o(\xi)\phi_j(\xi)\left(\frac{1}{|\xi|}\right)^{\alpha} e^{-\mathbf{i}\rho\ln(|\xi|)} d\xi.\end{aligned}$$

For every $j \in \mathbb{Z}$, integration by parts with respect to ξ implies

$$\left| \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x \cdot \xi} \delta_o(\xi) \phi_j(\xi) \left(\frac{1}{|\xi|}\right)^{\alpha} e^{-\mathbf{i}\rho \ln(|\xi|)} d\xi \right| \lesssim 2^{j(\mathbf{N}-\alpha)} \left(2^j |x|\right)^{-N} (1+\rho)^N \tag{3. 12}$$

for every $N \geq 1$. Indeed, every ∂_ξ acting on $\delta_o(\xi)\phi_j(\xi)\left(\frac{1}{|\xi|}\right)^{\alpha} e^{-\mathbf{i}\rho \ln(|\xi|)}$ gains a factor bounded by a constant multiple of $2^{-j} \max\{1, \rho\}$.

We choose

$$N = 0 \qquad \text{for} \qquad |x| \leq 2^{-j} \tag{3. 13}$$

and

$$N = \mathbf{N} \qquad \text{for} \qquad |x| > 2^{-j}. \tag{3. 14}$$

By summing over (3. 12) all $j \in \mathbb{Z}$, the estimate in (3. 11) implies

$$\begin{aligned} \left|\Omega_o^{\alpha+\mathbf{i}\rho}(x)\right| &\lesssim \sum_j \left| \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi \mathbf{i} x\cdot\xi} \delta_o(\xi)\phi_j(\xi)\left(\frac{1}{|\xi|}\right)^{\alpha} e^{-\mathbf{i}\rho\ln(|\xi|)} d\xi \right| \\ &\lesssim \sum_{|x|\leq 2^{-j}} 2^{j(\mathbf{N}-\alpha)} + (1+\rho)^{\mathbf{N}} \sum_{|x|>2^{-j}} 2^{j(\mathbf{N}-\alpha)} \left(2^j|x|\right)^{-\mathbf{N}} \\ &\lesssim (1+\rho)^{\mathbf{N}} \left(\frac{1}{|x|}\right)^{\mathbf{N}-\alpha}. \end{aligned} \tag{3. 15}$$

□

Define the n-parameters dilation

$$\mathbf{t}x = \left(2^{t_1}x_1, 2^{t_2}x_2, \ldots, 2^{t_n}x_n\right). \tag{3. 16}$$

Let $\mathbf{Q}^{\mathbf{t}}$ to be a dilated variant of $\mathbf{Q}$, such that $|\mathbf{Q}_i^{\mathbf{t}}|^{\frac{1}{\mathbf{N}_i}} = 2^{t_i}|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}$ for $i = 1, 2, \ldots, n$.

Suppose that $\omega, \sigma \in \mathbf{A}^{\alpha}_{pqr}$ in (1. 7) for $1 < p \leq q < \infty$ and $r > 1$. We have

$$\begin{aligned} &\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(\mathbf{t}x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(\mathbf{t}x)\,dx\right\}^{\frac{p-1}{pr}} \\ &= \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}} \\ &= \prod_{i=1}^n 2^{-t_i\left(\alpha_i - \frac{\mathbf{N}_i}{p} + \frac{\mathbf{N}_i}{q}\right)} \prod_{i=1}^n |\mathbf{Q}_i^{\mathbf{t}}|^{\frac{\alpha_i}{\mathbf{N}_i} - \left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}} \\ &\leq \prod_{i=1}^n 2^{-t_i\left(\alpha_i - \frac{\mathbf{N}_i}{p} + \frac{\mathbf{N}_i}{q}\right)} \mathbf{A}^{\alpha}_{pqr}(\omega, \sigma). \end{aligned} \tag{3. 17}$$

We now recall the weighted inequality for one-parameter fractional integrals proved by Sawyer and Wheeden [5]. For each $\mathbf{t}$ fixed, Theorem 1 in [5] implies

$$\begin{aligned}&\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}}|f(y)|\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}dy\right\}^{q}\omega^{q}(\mathbf{t}x)dx\right\}^{\frac{1}{q}}\\ &\lesssim\ \prod_{i=1}^{n}2^{-t_i\left(\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right)}\mathbf{A}^{\alpha}_{pqr}(\omega,\sigma)\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|f(x)\right|^{p}\sigma^{p}(\mathbf{t}x)dx\right\}^{\frac{1}{p}}\end{aligned}\tag{3. 18}$$

for $1<p\leq q<\infty$.

By carrying out the proof of the applied theorem, given in section 2 of [5], we find that the implied constant in (3. 18) depends only on p, q, r, α and $\mathbf{N}$.

Recall from (3. 8). We have

$$\begin{aligned}&\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|\left(f*\Omega^{\alpha+\mathbf{i}\rho}_{\mathbf{t}}\right)(x)\right|^{q}\omega^{q}(x)dx\right\}^{\frac{1}{q}}\\
&=\ \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|\int_{\mathbb{R}^{\mathbf{N}}}f(y)\left\{\int_{\mathbb{R}^{\mathbf{N}}}e^{2\pi\mathbf{i}(x-y)\cdot\xi}\delta_{\mathbf{t}}(\xi)\prod_{i=1}^{n}\left(\frac{1}{|\xi_i|}\right)^{\alpha_i+\mathbf{i}\rho_i}d\xi\right\}dy\right|^{q}\omega^{q}(x)dx\right\}^{\frac{1}{q}}\\
&=\ \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|\int_{\mathbb{R}^{\mathbf{N}}}f(\mathbf{t}y)\left\{\int_{\mathbb{R}^{\mathbf{N}}}e^{2\pi\mathbf{i}(x-y)\cdot\xi}\delta_{o}(\xi)\prod_{i=1}^{n}\left(\frac{1}{2^{-t_i}|\xi_i|}\right)^{\alpha_i+\mathbf{i}\rho_i}\prod_{i=1}^{n}2^{-t_i\mathbf{N}_i}d\xi\right\}\prod_{i=1}^{n}2^{t_i\mathbf{N}_i}dy\right|^{q}\omega^{q}(\mathbf{t}x)\prod_{i=1}^{n}2^{t_i\mathbf{N}_i}dx\right\}^{\frac{1}{q}}\\
&\lesssim\ \prod_{i=1}^{n}2^{t_i\left(\alpha_i+\frac{\mathbf{N}_i}{q}\right)}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|f(\mathbf{t}y)\right|\left|\int_{\mathbb{R}^{\mathbf{N}}}e^{2\pi\mathbf{i}(x-y)\cdot\xi}\delta_{o}(\xi)\prod_{i=1}^{n}\left(\frac{1}{|\xi_i|}\right)^{\alpha_i+\mathbf{i}\rho_i}d\xi\right|dy\right\}^{q}\omega^{q}(\mathbf{t}x)\,dx\right\}^{\frac{1}{q}}\\
&\lesssim\ (1+\rho)^{\mathbf{N}}\prod_{i=1}^{n}2^{t_i\left(\alpha_i+\frac{\mathbf{N}_i}{q}\right)}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|f(\mathbf{t}y)\right|\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}dy\right\}^{q}\omega^{q}(\mathbf{t}x)\,dx\right\}^{\frac{1}{q}}\qquad\text{by (3. 9)}\\
&\lesssim\ (1+\rho)^{\mathbf{N}}\prod_{i=1}^{n}2^{t_i\left(\alpha_i+\frac{\mathbf{N}_i}{q}\right)}2^{-t_i\left(\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right)}\mathbf{A}^{\alpha}_{pqr}(\omega,\sigma)\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|\left(f\sigma\right)(\mathbf{t}x)\right|^{p}dx\right\}^{\frac{1}{p}}\qquad\text{by (3. 17)-(3. 18)}\\
&=\ (1+\rho)^{\mathbf{N}}\,\mathbf{A}^{\alpha}_{pqr}(\omega,\sigma)\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left|\left(f\sigma\right)(x)\right|^{p}dx\right\}^{\frac{1}{p}}.\end{aligned}\tag{3. 19}$$

In particular, we consider $\rho=\rho_i=0,\ i=1,2,\ldots,n$ in (3. 19). Observe that for each $\mathbf{t}$ fixed, $\Delta_{\mathbf{t}}\mathbf{I}_\alpha$ is essentially an one-parameter fractional integral operator, satisfying

$$\left\|\omega\left(\Delta_{\mathbf{t}}\mathbf{I}_\alpha\right)\sigma^{-1}\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})\longrightarrow\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})}\ \lesssim\ \mathbf{A}^{\alpha}_{pqr}(\omega,\sigma).\tag{3. 20}$$

4 Eccentricity Decaying and Interpolations

Recall $\delta_{\mathbf{t}}$ from (3. 2) which is supported on the Dyadic cone $\Lambda_{\mathbf{t}}$ in (3. 3). For each $\mathbf{t}$ fixed, the cone $\Lambda_{\mathbf{t}}$ intersects at most 3^n of others, with similar eccentricity, denoted by $\Lambda_{\mathbf{s}}$ whereas $|t_i - s_i| \leq 1, i = 1,2,\ldots,n$. The partial sum operator $\Delta_{\mathbf{t}}$ defined in (3. 4) commutes with convolution operators. Let $\Omega_{\mathbf{t}}^{\alpha}$ to be given in (3. 7). We have

$$f * \Omega_{\mathbf{t}}^{\alpha} = \sum_{\mathbf{s}} \Delta_{\mathbf{s}}\Big(f * \Omega_{\mathbf{t}}^{\alpha}\Big) = \left\{ \sum_{|t_i - s_i| \leq 1} \Delta_{\mathbf{s}} f \right\} * \Omega_{\mathbf{t}}^{\alpha}. \tag{4. 1}$$

Suppose that f is a radially symmetric function. Its Fourier transform

$$\widehat{f}(\xi) = \left(\frac{1}{|\xi|}\right)^{\frac{\mathbf{N}-2}{2}} \int_0^{\infty} \mathcal{J}_{\frac{\mathbf{N}-2}{2}} \left(|x||\xi|\right) f(|x|)|x|^{\frac{\mathbf{N}-2}{2}} |x| d|x| \tag{4. 2}$$

is another, where $\mathcal{J}_{\frac{\mathbf{N}-2}{2}}$ is Bessel function of order $(\mathbf{N}-2)/2$. By applying Plancherel theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^{\mathbf{N}}} \left|\Big(\Delta_{\mathbf{t}} f\Big)(x)\right|^2 dx &= \int_{\mathbb{R}^{\mathbf{N}}} \left|\delta_{\mathbf{t}}(\xi)\widehat{f}(\xi)\right|^2 d\xi \\ &\lesssim \int_{\Lambda_{\mathbf{t}}} \left|\widehat{f}(\xi)\right|^2 d\xi \qquad \Lambda_{\mathbf{t}} \text{ defined in (3. 3)} \\ &\lesssim \prod_{i=1}^{n} 2^{-t_i} \int_{\mathbb{R}^{\mathbf{N}}} \left|\widehat{f}(\xi)\right|^2 d\xi = \prod_{i=1}^{n} 2^{-t_i} \int_{\mathbb{R}^{\mathbf{N}}} \left|f(x)\right|^2 dx. \end{aligned} \tag{4. 3}$$

Notice that $\widehat{f}$ is radially symmetric and the range of $\arccos\left(\frac{|\xi_i|}{|\xi|}\right)$ for $\xi \in \Lambda_{\mathbf{t}}$ defined in (3. 3) is bounded by a constant multiple of 2^{-t_i} for every $i = 1,2,\ldots,n$.

Suppose that

$$\frac{\alpha_i}{\mathbf{N}_i} = \frac{1}{p} - \frac{1}{q}, \qquad i = 1,2,\ldots,n. \tag{4. 4}$$

Let $p = 2$ in (4. 4). The estimates in (3. 19) and (4. 3) imply

$$\begin{aligned} \left\{ \int_{\mathbb{R}^{\mathbf{N}}} \left|\Big(\Delta_{\mathbf{t}} f * \Omega_{\mathbf{t}}^{\alpha+\mathbf{i}\rho}\Big)(x)\right|^q dx \right\}^{\frac{1}{q}} &\lesssim (1+\rho)^{\mathbf{N}} \left\{ \int_{\mathbb{R}^{\mathbf{N}}} \left|\Big(\Delta_{\mathbf{t}} f\Big)(x)\right|^2 dx \right\}^{\frac{1}{2}} \\ &\lesssim (1+\rho)^{\mathbf{N}} \prod_{i=1}^{n} 2^{-\frac{t_i}{2}} \left\{ \int_{\mathbb{R}^{\mathbf{N}}} \left|f(x)\right|^2 dx \right\}^{\frac{1}{2}}. \end{aligned} \tag{4. 5}$$

By using the estimates in (4. 1) and (4. 5), together with Minkowski inequality, we have

$$\left\{ \int_{\mathbb{R}^{\mathbf{N}}} \left|\Big(f * \Omega_{\mathbf{t}}^{\alpha+\mathbf{i}\rho}\Big)(x)\right|^q dx \right\}^{\frac{1}{q}} \lesssim (1+\rho)^{\mathbf{N}} \prod_{i=1}^{n} 2^{-\frac{t_i}{2}} \left\{ \int_{\mathbb{R}^{\mathbf{N}}} \left|f(x)\right|^2 dx \right\}^{\frac{1}{2}} \tag{4. 6}$$

for every $\mathbf{t}$.

Let $1 < p_i < q_i < \infty$, $i = 1, 2$ satisfying (4. 4). The estimate in (3. 19) directly implies that $\Delta_{\mathbf{t}}\mathbf{I}_\alpha \colon \mathbf{L}^{p_i}\left(\mathbb{R}^{\mathbf{N}}\right) \longrightarrow \mathbf{L}^{q_i}\left(\mathbb{R}^{\mathbf{N}}\right)$, $i = 1, 2$. Moreover, we set $p_2 = 2$. Consider $p \in (1, 2] \cup [2, \infty)$ where (4. 4) holds for $1 < p < q < \infty$. We can find p_1, q_1 such that $1/p = (1-t)/p_1 + t/2$ and $1/q = (1-t)/q_1 + t/q_2$ for some $0 < t < 1$. By applying Riesz interpolation theorem [1], together with the estimate in (4. 5), we have

$$\left\{\int_{\mathbb{R}^{\mathbf{N}}} \left|\left(f * \Omega_{\mathbf{t}}^{\alpha+\mathbf{i}\rho}\right)(x)\right|^q dx\right\}^{\frac{1}{q}} \lesssim (1+\rho)^{\mathbf{N}} \prod_{i=1}^n 2^{-\varepsilon t_i} \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left|f(x)\right|^p dx\right\}^{\frac{1}{p}} \tag{4. 7}$$

for some $\varepsilon = \varepsilon(p) > 0$ and every $\mathbf{t}$.

Let $z = \lambda + \mathbf{i}\mu \in \mathbb{C}$ where $0 < \lambda < 1$. Recall the multi-indices γ and β respectively defined in (2. 6) and (2. 8). Consider

$$\left(\mathbf{U}_{\mathbf{t}}^z f\right)(x) \doteq \left(f * \Omega_{\mathbf{t}}^z\right)(x) \tag{4. 8}$$

for every $\mathbf{t}$, where

$$\Omega_{\mathbf{t}}^z(x) = \int_{\mathbb{R}^{\mathbf{N}}} e^{2\pi\mathbf{i}x\cdot\xi}\delta_{\mathbf{t}}(\xi) \prod_{i=1}^n \left(\frac{1}{|\xi_i|}\right)^{\beta_i(1-z)+\gamma_i z} d\xi. \tag{4. 9}$$

Let $\omega, \sigma \in \mathbf{A}_{pqr}^{\alpha}$ in (1. 7) for $1 < p < q < \infty$ and some $r > 1$. Recall the implication from (2. 10). The estimate in (3. 19) implies

$$\left\|\omega^{s(\lambda+\mathbf{i}\mu)}\mathbf{U}_{\mathbf{t}}^{\lambda+\mathbf{i}\mu}\sigma^{-s(\lambda+\mathbf{i}\mu)}f\right\|_{\mathbf{L}^q\left(\mathbb{R}^{\mathbf{N}}\right)} \lesssim (1+(\beta+\gamma)\mu)^{\mathbf{N}}\, \mathbf{A}_{pqt}^{(1-\lambda)\beta+\lambda\gamma}\left(\omega^s, \sigma^s\right) \left\|f\right\|_{\mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)} \tag{4. 10}$$

provided that $\omega^{s\lambda}, \sigma^{s\lambda} \in \mathbf{A}_{pqt}^{(1-\lambda)\beta+\lambda\gamma}$ for $r = st$ and $0 < \lambda < 1$.

Observe that $\omega^{s(\lambda+\mathbf{i}\mu)}\mathbf{U}_{\mathbf{t}}^{\lambda+\mathbf{i}\mu}\sigma^{-s(\lambda+\mathbf{i}\mu)}$ in (4. 10) is an analytic family of operators satisfying the *admissible growth* required in Stein [2] for every $0 < \lambda < 1$.

Furthermore, recall the implication from (2. 7). For $\lambda = 1$, the estimate in (3. 19) implies

$$\left\|\omega^{s(1+\mathbf{i}\mu)}\mathbf{U}_{\mathbf{t}}^{1+\mathbf{i}\mu}\sigma^{-s(1+\mathbf{i}\mu)}f\right\|_{\mathbf{L}^q\left(\mathbb{R}^{\mathbf{N}}\right)} \lesssim (1+(\beta+\gamma)\mu)^{\mathbf{N}}\, \mathbf{A}_{pqt}^{\gamma}\left(\omega^s, \sigma^s\right) \left\|f\right\|_{\mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)} \tag{4. 11}$$

provided that $\omega^s, \sigma^s \in \mathbf{A}_{pqt}^{\gamma}$ for $r = st$.

On the other hand, for $\lambda = 0$, the estimate in (4. 7) implies

$$\left\|\omega^{\mathbf{i}s\mu}\mathbf{U}_{\mathbf{t}}^{\mathbf{i}\mu}\sigma^{-\mathbf{i}s\mu}f\right\|_{\mathbf{L}^q\left(\mathbb{R}^{\mathbf{N}}\right)} \lesssim (1+(\beta+\gamma)\mu)^{\mathbf{N}} \prod_{i=1}^n 2^{-\varepsilon t_i} \left\|f\right\|_{\mathbf{L}^p\left(\mathbb{R}^{\mathbf{N}}\right)} \tag{4. 12}$$

for some $\varepsilon > 0$.

Let $0 < \vartheta = s^{-1} < 1$ where $s > 1$. We have

$$\begin{aligned} &(1-\vartheta)\,\beta_i + \vartheta\gamma_i \\ &= \left(1-\frac{1}{s}\right)\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) + \left(\frac{1}{s}\right)\left[s\alpha_i - (s-1)\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)\right] \\ &= \left(1-\frac{1}{s}\right)\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) + \alpha_i - \left(1-\frac{1}{s}\right)\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) \\ &= \alpha_i, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{4. 13}$$

Recall the estimates given in (4. 10) and (4. 11)-(4. 12). Moreover, from (4. 13) we have $\omega^{r\vartheta}\mathbf{U}^{\vartheta}_{\mathbf{t}}\sigma^{-r\vartheta} = \omega(\Delta_{\mathbf{t}}\mathbf{I}_\alpha)\sigma^{-1}$. By applying Stein interpolation theorem for the analytic operators, stated as Theorem 1 in Stein [2], we have

$$\left\|\omega\Big(\Delta_{\mathbf{t}}\mathbf{I}_\alpha\Big)f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \lesssim \Big(\mathbf{A}^{\gamma}_{pqt}\left(\omega^s,\sigma^s\right)\Big)^{\frac{1}{s}}\prod_{i=1}^{n}2^{-\varepsilon\left(\frac{s-1}{s}\right)t_i}\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \tag{4. 14}$$

for $1 < p < q < \infty$.

Recall from (2. 7) where $\mathbf{A}^{\gamma}_{pqt}\left(\omega^s,\sigma^s\right) = \Big(\mathbf{A}^{\alpha}_{pqr}\left(\omega,\sigma\right)\Big)^s$ for $r = st$. By using the estimate in (4. 14) and applying Minkowski inequality, we obtain

$$\left\|\omega\mathbf{I}_\alpha f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \lesssim \mathbf{A}^{\alpha}_{pqr}(\omega,\sigma)\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}. \tag{4. 15}$$

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email: zipeng.wang@cantab.net